Mathematical analysis of a three component chemotactic system

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The biological context

Purpose : Explain patterns arising in experiments with strains of $E - Coli$ bacteria

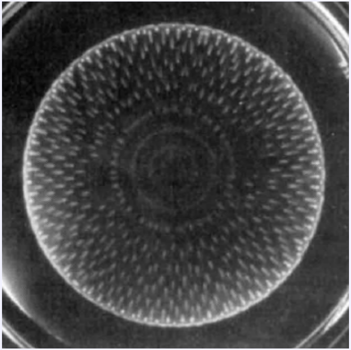
Budrene and Berg have performed experiments showing that chemotactic strains of bacteria *E. coli*, inoculated in semi-solid agar, form stable and remarkably complex but geometrically regulated spatial patterns such as swarm rings, radial spots, and interdigitated arrays of spots.

A purpose of Mayan Mimura and his group has been to propose mathematical models to reproduce these patterns.

E. O. Budrene, H. Berg, *Dynamics of formation of symmetrical patterns by chemotactic bacteria*, Nature **376** (1995), 49–53.

A. Aotani, M. Mimura, T. Mollee, *A model aided understanding of spot pattern formation in chemotactic E. coli colonies*, Japan J. Indust. Appl. Math. **27** (2010), 5–22.

An experimental picture



Experimental chevron pattern
(by courtesy of Budrene and Berg, 1995)

Mayan Mimura has built two successive models in order to be able to reproduce such patterns.

I am going to show them to you.

A two component chemotaxis model

Mimura and Tsujikawa have first proposed the following mesoscopic model based on the chemotaxis and growth of bacteria:

$$
\begin{aligned}
u_t &= d_u \Delta u - div(u \nabla \chi(c)) + f(u) \\
c_t &= d_c \Delta c + \alpha u - \beta c.
\end{aligned}
$$

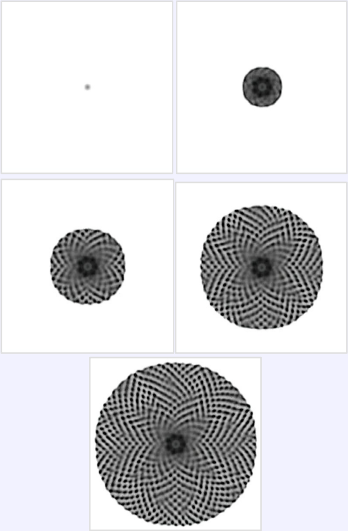
Here, $u = u(x,t)$ denotes the density of cells and $c = c(x,t)$ is the concentration of a chemo-attractant. The constants d_u, d_c, α, and β are supposed to be positive, χ is the sensitive function of chemotaxis and $f(u)$ is a growth function with an Allee effect. In the absence of the function $f(u)$, this system reduces to the Keller-Segel model.

Because of the growth term, the corresponding problem with homogeneous Neumann boundary conditions possesses a global in time solution.

A three component model

$$
\begin{aligned}
u_t &= d_u \Delta u - div(u \nabla \chi(c)) + g(u)nu - b(n)u, \\
c_t &= d_c \Delta c + \alpha u - \beta c, \\
n_t &= d_n \Delta n - \gamma g(u)nu.
\end{aligned}
$$

Numerical simulations



Formation of the chevron pattern

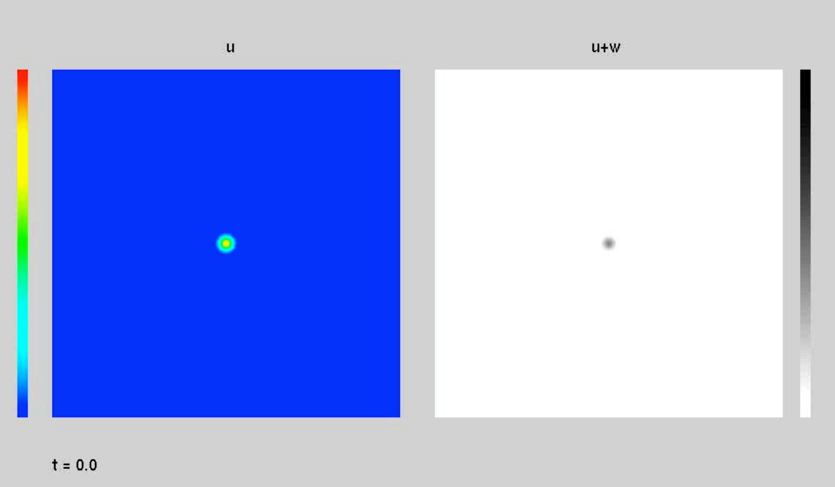
Introducing an inactive bacteria

But then Mimura introduced, next to the density u of active bacteria, the density w of inactive ones

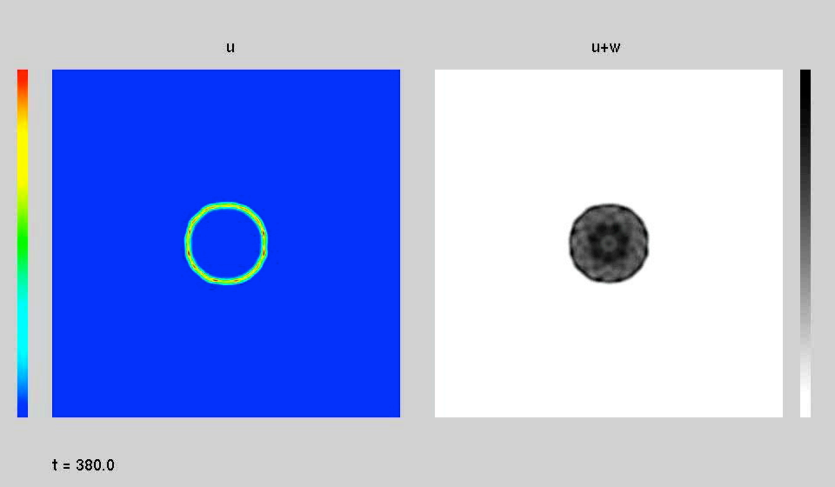
$$\begin{aligned}
u_t &= \Delta u - div(u\nabla\chi(c)) + g(u)nu - b(n)u, \\
c_t &= d_c\Delta c + \alpha u - \beta c, \\
n_t &= d_n\Delta n - \gamma g(u)nu, \\
w_t &= b(n)u.
\end{aligned}$$

Note that the knowledge of u and n is necessary to derive w, whereas the coupling is incomplete since the knowledge of w is not needed for the computation of u and n. One can visualize w as a sort of memory term, which takes into account the values of the product $b(n)u$ at all previous times.

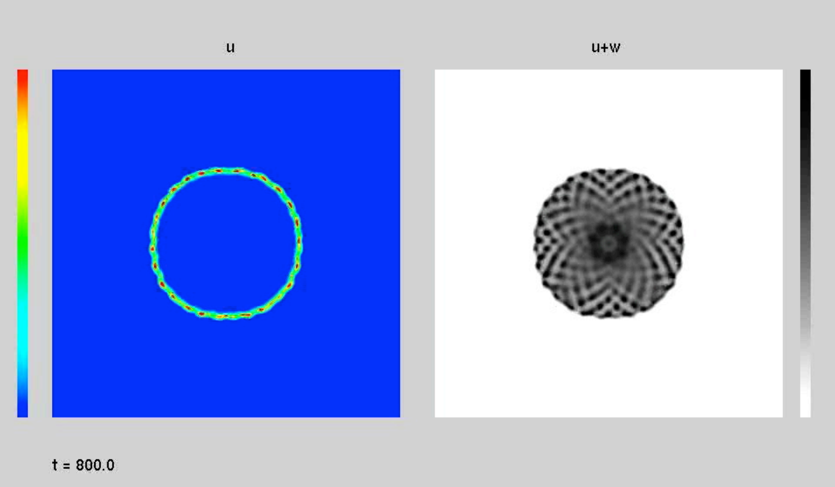
Time evolution of active and inactive bacteria



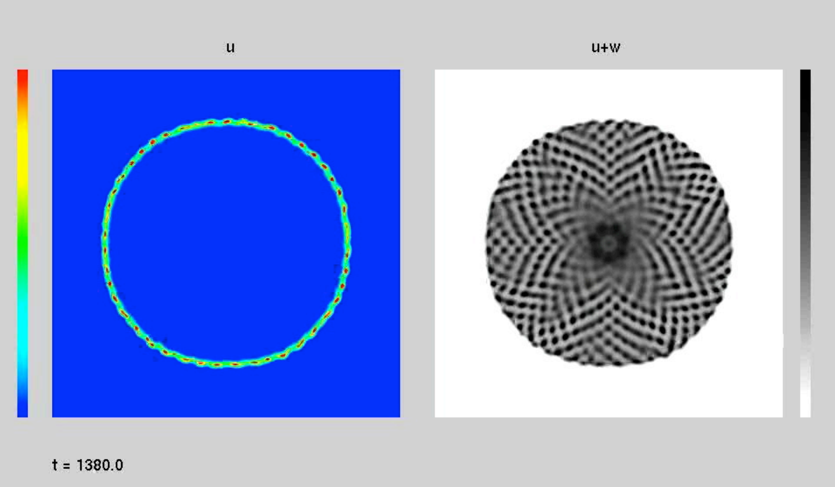
Time evolution of active and inactive bacteria



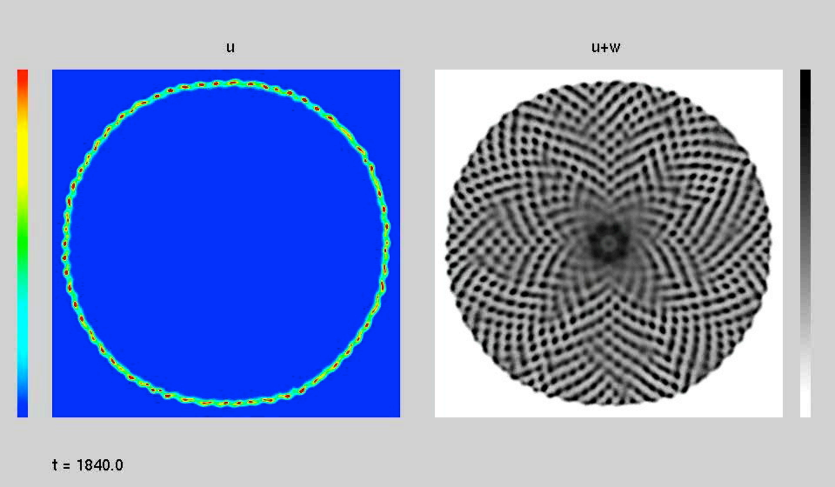
Time evolution of active and inactive bacteria



Time evolution of active and inactive bacteria



Time evolution of active and inactive bacteria



Already known mathematical results

The existence and uniqueness of the solution on the time interval $(0, +\infty)$ has been proved by

P. P. Htoo, M. Mimura, I. Takagi, *Global solutions to a one-dimensional nonlinear parabolic system modeling colonial formation by chemotactic bacteria*, Adv. Stud. Pure Math. **47-2** (2007), 613–622,

in the case of one space dimension and for a special choice of the functions χ, *g* and *n*.

The full problem

We study the equations

$$\begin{aligned} u_t &= \Delta u - div(u \nabla \chi(c)) + g(u)nu - b(n)u \\ c_t &= d_c \Delta c + \alpha u - \beta c \\ n_t &= d_n \Delta n - \gamma g(u)nu \\ w_t &= b(n)u, \end{aligned}$$

together with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \qquad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0$$

and the nonnegative initial conditions

$$u(.,0) = u_0,\ c(.,0) = c_0,\ n(.,0) = n_0,\ w(.,0) = w_0.$$

Mathematical assumptions

The diffusion coefficients $d_c > 0$ and $d_n > 0$ as well as the coefficients $\alpha > 0$, $\beta > 0$, $\gamma > 0$ denote given constants. Moreover, we impose the following assumptions on the functions $g, b, \chi \in C^1([0, \infty))$:
(i) $g(0) = 0$ and $g = g(s)$ is increasing for $s > 0$ and bounded from above by a positive constant G_0;
(ii) $b(0) = B_0 > 0$ and $b = b(s)$ is decreasing for $s > 0$ and positive;
(iii) $\chi' \in L^\infty([0, \infty))$.

Space homogeneous solutions

Note that if the initial conditions are independent of x, namely, if

$$u_0(x) = \overline{u_0}, \quad c_0(x) = \overline{c}_0, \quad n_0(x) = \overline{n_0}, \quad w_0(x) = \overline{w}_0,$$

for certain constants $\overline{u}_0, \overline{c}_0, \overline{n}_0, \overline{w}_0 \in [0, \infty)$, then the corresponding solution is also independent of x.

For every nonnegative, constant initial condition, the corresponding solution

$$\big(\bar{u}(t), \bar{c}(t), \bar{n}(t), \bar{w}(t)\big)$$

is global in time and converges exponentially fast towards the constant vector $(0, 0, \bar{n}_\infty, \bar{w}_\infty)$ for some $\bar{n}_\infty \geq 0$ and $\bar{w}_\infty \geq 0$ depending on the initial conditions.

This result is proved by analyzing the phase portrait of the corresponding system of ordinary differential equations.

Large time behavior of the integrals

Assume that a nonnegative solution (u, c, n, w) exists for all $t > 0$. Then

$$\int_\Omega u(x,t)dx \to 0 \quad \text{and} \quad \int_\Omega c(x,t)dx \to 0 \qquad \text{as} \;\; t \to \infty,$$

and there are constants $n_\infty > 0$ and $w_\infty > 0$ such that

$$\int_\Omega n(x,t)dx \to n_\infty \quad \text{and} \quad \int_\Omega w(x,t)dx \to w_\infty \qquad \text{as} \;\; t \to \infty.$$

One dimensional solutions

Assume that $d = 1$ and $\Omega \subset \mathbb{R}$ is an open and bounded interval. For every initial condition $u_0, n_0, w_0 \in L^\infty(\Omega)$ and $c_0 \in W^{1,\infty}(\Omega)$, the corresponding solution (u, c, n, w) exists for all $t > 0$. Moreover, there exists a constant $n_\infty \geq 0$ and a nonnegative function $w_\infty \in L^\infty(\Omega)$ such that

$$\big(u(x,t), c(x,t), n(x,t), w(x,t)\big) \to (0, 0, n_\infty, w_\infty(x))$$

as $t \to \infty$ exponentially fast in $L^\infty(\Omega)$.

This result extends that of P. P. Htoo, M. Mimura, I. Takagi to the case of more general nonlinear functions.

Existence and uniqueness of a local in time solution

Following the book by Yagi, we introduce the following spaces

$$
\begin{aligned}
H^2_N(\Omega) &= \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\}, \\
\mathcal{H}^4_{N^2}(\Omega) &= \{u \in H^2_N(\Omega) : \Delta u \in H^2_N(\Omega)\}.
\end{aligned}
$$

One can deduce as in the book by Yagi that for every initial datum $(u_0, c_0, n_0, w_0) \in L^2(\Omega) \times H^2_N(\Omega) \times L^2(\Omega) \times L^\infty(\Omega)$, there exists $T > 0$ such that the problem possesses a unique local in time solution satisfying

$$
\begin{aligned}
& u \in C((0,T]; H^2_N(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)), \\
& c \in C((0,T]; \mathcal{H}^4_{N^2}(\Omega)) \cap C([0,T]; H^2_N(\Omega)) \cap C^1((0,T]; H^2_N(\Omega)), \\
& n \in C((0,T]; H^2_N(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)), \\
& w \in C^1([0,T]; L^\infty(\Omega)).
\end{aligned}
$$

Nonnegativity and upper bound for n

If $u_0 \geq 0$, $c_0 \geq 0$, $n_0 \geq 0$, $w_0 \geq 0$ almost everywhere, then the local-in-time solution is such that

$$u(.,t) \geq 0, c(.,t) \geq 0, n(.,t) \geq 0, w(.,t) \geq 0,$$

for all $0 < t < T$.

Since $\gamma g(u)u \geq 0$ and $n_0 \geq 0$, the maximum principle also implies that

$$0 \leq n(x,t) \leq \|n_0\|_\infty \quad \text{for all} \quad x \in \Omega,\ t \in [0,T].$$

Smallness assumption on the initial data

Let $d \in \{2, 3\}$ and (u, c, n, w) be a nonnegative local in time solution. Fix $p_0 \in (\frac{d}{2}, \frac{d}{d-2})$. There exists $\varepsilon > 0$ such that if

$$\max(\|u_0\|_{p_0}, \|n_0\|_1, \|\nabla c_0\|_{2p_0}) < \varepsilon,$$

then the solution (u, c, n, w) exist for all $t > 0$ and satisfies

$$\sup_{t>0} \|u(t)\|_\infty < \infty.$$

Moreover, there exists a constant $n_\infty \geq 0$ and a nonnegative function $w_\infty \in L^\infty(\Omega)$ such that

$$\big(u(x,t), c(x,t), n(x,t), w(x,t)\big) \to (0, 0, n_\infty, w_\infty(x))$$

as $t \to \infty$ exponentially fast in $L^\infty(\Omega)$.

Global existence for suitable sensitivity functions

Let (u, c, n, w) be a nonnegative local-in-time solution. Assume that the chemotactic sensitivity function satisfies

$$\frac{d\chi(s)}{ds} \leq \frac{\chi_0}{(1+\bar{c}s)^k} \qquad \text{for all} \quad s \geq 0$$

for some constants $\chi_0 > 0$, $k > 1$ and $\bar{c} > 0$. Then the solution (u, c, n, w) exists globally in time and satisfies $\|u(t)\|_\infty < \infty$. Moreover, there exist a constant $n_\infty \geq 0$ and a nonnegative function $w_\infty \in L^\infty(\Omega)$ such that

$$\big(u(x,t), c(x,t), n(x,t), w(x,t)\big) \to \big(0, 0, n_\infty, w_\infty(x)\big)$$

as $t \to \infty$ exponentially fast in $L^\infty(\Omega)$.

Blow up in finite time?

We prove a result which could go in this direction, however with slightly modifying the equations, with a linear sensitivity function and an elliptic equation for the concentration of the chemo-attractant.

$$
\begin{aligned}
u_t &= \Delta u - \chi_0 div(u \nabla c) + g(u)nu - b(n)u \\
0 &= \Delta c + u - c \\
n_t &= \Delta n - \gamma g(u)nu
\end{aligned}
$$

in a bounded domain $\Omega \subset \mathbb{R}^2$, supplemented with the Neumann boundary conditions

$$
\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0,
$$

and with nonnegative initial functions

$$
u(x,0) = u_0(x), \quad n(x,0) = n_0(x).
$$

Blow up in finite time?

Let $d = 2$ and let $q \in \Omega$. Assume that $\chi_0 > 0$. Consider a local-in-time nonnegative solution (u, c, n) of the modified problem above and assume that

$$M_0 = \int_\Omega u_0(x)dx > \frac{8\pi}{\chi_0}.$$

If $\int_\Omega u_0(x)|x-q|^2dx$ is sufficiently small, then the solution (u, c, n) cannot be extended to a global one.

Duhamel's formula

$$u(t) = \mathrm{e}^{\Delta t} u_0 + \int_0^t \nabla \cdot \mathrm{e}^{\Delta(t-s)} u(s) \nabla \chi(c(s)) ds$$
$$+ \int_0^t \mathrm{e}^{\Delta(t-s)} u(s)(g(u)n - b(n))(s) ds,$$

Estimates for the solution of the heat equation

$$\|e^{t\Delta}f\|_{L^p(\Omega)} \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}e^{-\lambda_1 t}\|f\|_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$ satisfying $\int_\Omega f d\mathbf{x} = 0$ and all $t > 0$;

$$\|e^{t\Delta}f\|_{L^p(\Omega)} \le C\left(1 + t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\|f\|_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$ and all $t > 0$;

$$\|\nabla e^{t\Delta}f\|_{L^p(\Omega)} \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}e^{-\lambda_1 t}\|f\|_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$ and all $t > 0$.

I thank you for your attention