PREDATORS-PREY MODELS WITH COMPETITION: EXISTENCE, BIFURCATION AND QUALITATIVE PROPERTIES

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Abstract. We consider a model that we proposed to study of environments populated by both preys and predators, with the possibility for the predators to actively compete for the territory. For this model we study existence and uniqueness of solutions, and the asymptotic properties in time, showing that the solutions have different behavior depending on the choice of the parameters. We also construct heterogeneous stationary solutions and study their behavior in some singular limit, we then use these informations to study some properties such as the existence of the solution that maximizes the total population of predators, which in some circumstances may contain more than one group of competing predators.

1. Introduction

In a recent paper [BZa] we have proposed a model to describe the emergence of territoriality in predatory animals: specifically, our aim was to show if aggressiveness among predators is a key mechanism in order to explain territorial behaviors. To accomplish this, we consider a given region $\Omega \subset \mathbb{R}^n$ with $n \leq 2$ (the restriction on the dimension is purely for modeling reasons) occupied by $N+1$ densities, where one of the densities, referred to by the letter u, is composed of preys, while the other N densities, denoted by the symbols $w_1, \dots, w_N$, are predators. Each of the densities evolves in time following a natural law of Lotka-Volterra type. As a result, the model proposed in [BZa] is synthesized in the system

$$\begin{cases} w_{i,t} - d_i \Delta w_i = \left(-\omega_i + k_i u - \mu_i w_i - \beta \sum_{j \neq i} a_{ij} w_j\right) w_i \\ u_t - D\Delta u = \left(\lambda - \mu u - \sum_{i=1}^N k_i w_i\right) u \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega \times (0, +\infty)$, completed by homogeneous Neumann boundary and smooth initial conditions. The parameters of the model are easily explained: some terms in the equations model internal mechanism in the populations

- $D, d_1, \dots, d_N$ are the diffusion rates of the different populations, and thus are always considered positive in the following;
- $\lambda > 0$ is the reproductivity coefficient of the preys, and $\mu \geq 0$ stands for the possible saturability of the environment due to an excess of preys;
- ω_i is the death coefficient of the predators that takes into account the starvation caused by the absence of the prey u, and μ_i takes into account possible saturation phenomena in the predator populations (for instance, an internal competition between member of the same density; typically we will not consider this term relevant in the following).

The other terms are, on the other hand, responsible for the interaction between different densities

- $k_1, \dots, k_N$ govern the predation rates, that is the success of the predator w_i in catching the prey u over the probability of an encounter;
- the elements $a_{ij} > 0$ represent how the presence of the density w_j affects the density w_i, and the particular choice of the sign suggest that we only consider competing interactions. The parameter $\beta \geq 0$ on the other hand expresses the strength of the interaction: higher values of β correspond to more aggressive predators.

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Similar models have already been introduced in the ecological and mathematical literature, starting from the seminal paper by Volterra [Vol28] on predator-preys interactions, to the more recent contribution [DD94] that started the study of strongly interaction systems of elliptic equations (in that case, modeling populations of competing predators), the study on the evolution of dispersal by means of systems of many interacting predators [DHMP98], and the papers on the qualitative properties of the solutions to such systems [CTV05a, CKL09, DWZ12]. The novelty in our model, that complicates the analysis but allows for more profound results, is the inclusion in the equation of the equation for the preys and of the competition between the predators. A more in depth comparison with the results in the scientific literature can be found in [BZa], to which we refer the interested reader.

The main results in this paper regarding the model (1.1) are summarized as follows (see also the following sections for more general statements of the results). For sufficiently smooth and positive initial data, the system (1.1) admits a unique, bounded and smooth solution, defined for all $t \geq 0$ (see Lemma 2.1). The competition is a driving force in the heterogeneity of the set of solutions, indeed the set of stationary solutions of the system is collapse to the set of constants if β is small (see Proposition 2.2), and is very rich for β large (see for instance Theorems 3.8, 3.11 and 3.12), but under some assumptions the asymptotic behavior can be described accurately (see Lemma 2.1 and Proposition 2.2). The solutions of the stationary system are regular, independently of the strength of the competition, and they converge to segregated configurations when $\beta \to +\infty$ (see Propositions 3.1 and 4.1), and this allows us to define also solutions to (1.1) in the case $\beta = +\infty$. Aggressiveness may help to resist and invasion by a foreign group, without any specific request on the parameters of the invader (see Propositions 2.4 and 2.8).

On the other hand, the strong competition limits the number densities that can survive in a given domain, indeed we have

Theorem (see Theorem 4.5)**.** *For a given smooth domain $\Omega \subset \mathbb{R}^N$, under some uniform-in-$N$ assumptions on the values of the parameters in the model, there exist $\bar{N} \in \mathbb{N}$ and $\bar{\beta} > 0$ such if $k > \bar{N}$ and $\beta > \bar{\beta}$ then the set of all non negative solutions to* (1.1) *contains only solutions of the form*

$$\|u_\beta - \lambda/\mu\|_{C^{2,\alpha}(\Omega)} + \|(w_{1,\beta}, \dots, w_{N,\beta})\|_{\mathcal{C}^{0,\alpha}(\Omega)} = o_\beta(1)$$

for every $\alpha \in (0,1)$.

And finally, we can show that, under some assumptions on the coefficients, there exist $N \in \mathbb{N}_0$ and a solution $(w_1, \dots, w_N, u)$ of (1.1) which maximizes the total population of predators, that is

Theorem (see Theorems 4.6 and 4.10)**.** *For any given smooth domain Ω, under some uniform-in-N assumptions on the values of the parameters in the model, there exists a number $\bar{N} \in \mathbb{N}$ and a solution $(w_1, \dots, w_N, u)$ of* (1.1) *with $\bar{N}+1$ non trivial components (possibly with $\beta = +\infty$) that maximizes the functional*

$$P(w_1, \dots, w_N, u) = \int_\Omega \sum_{i=1}^N w_i$$

among the set of all non negative solutions of (1.1)*. Moreover, if $\Omega = (a,b) \subset \mathbb{R}$ and μ is sufficiently small, the maximum is attained by a solution with more than one component of predators, that is, $\bar{N} \geq 2$.*

An immediate consequence of this is that competition between predators can be beneficial not only for the preys, but also for the predators themselves.

Structure of the paper. The paper is structured as follows: in Section 2 we consider some basic properties of the system, such as existence and regularity of solutions, together with some asymptotic properties of the system, concentrating ourself on the stability properties of specific solutions. In Section 3, thanks to a bifurcation analysis, we will show that the set of stationary solutions is very rich, and we also give a precise description of the solutions for large competition. Finally, in Section 4 we investigate some properties of the system with a large number of components, and we show in particular the existence of solutions which maximize the integral of the densities w_i.

2. General properties of the solutions

In this section we investigate some basic properties of the system: in particular we establish existence and uniqueness results for the solutions, we analysis the long time behavior of the set of solutions, and we also consider some stability properties of a special class of solutions, those that correspond to the case of only one predator and one prey. We recall the system

$$\begin{cases} w_{i,t} - d_i \Delta w_i = \left(-\omega_i + k_i u - \mu_i w_i - \beta \sum_{j \neq i} a_{ij} w_j \right) w_i \\ u_t - D\Delta u = \left(\lambda - \mu u - \sum_{i=1}^N k_i w_i \right) u \end{cases} \tag{2.1a}$$

in the domain $Q := \Omega \times (0, \infty)$, with $\Omega \Subset \mathbb{R}^n$ open, smooth, bounded and connected, completed by boundary and smooth initial conditions

$$\begin{cases} \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ w_i(x, 0) = w_i^0(x) \geq 0, u(x, 0) = u^0(x) \geq 0 & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.1b}$$

We start with the following existence result

Lemma 2.1. *Let $(w_1^0, \dots, w_N^0, u^0) \in \mathcal{C}^{0,\alpha}(\Omega)$ be a non-negative initial condition for the system (2.1). There exists a unique solution $(w_1, \dots, w_N, u) \in \mathcal{C}_x^{2,\alpha} \mathcal{C}_t^{1,\alpha/2}(Q)$ for all $\alpha \in (0, 1)$ which is defined for all $t > 0$; moreover the solution is bounded in $L^\infty(Q)$ and for any $\varepsilon > 0$ there exists $T = T_\varepsilon > 0$ such that*

$$\sup_{(x,t) \in \Omega \times [T, +\infty)} u(x, t) \leq \frac{\lambda}{\mu} + \varepsilon$$

$$\sup_{(x,t) \in \Omega \times [T, +\infty)} w_i(x, t) \leq \frac{\lambda k_i - \mu \omega_i}{\mu \mu_i} + \varepsilon.$$

Consequently, if there exists and index $i \in \{1, \dots, N\}$ such that $\lambda k_i \leq \mu \omega_i$, then

$$\lim_{t \to +\infty} \sup_{x \in \Omega} w_i(x, t) = 0.$$

As a result, in the following we shall also assume, for simplicity, that

(H) the relation $\lambda k_i > \mu \omega_i$ holds for any $i = 1, \dots, N$.

Proof. The existence of the solutions for $t \in [0, t_0]$ with $t_0 > 0$ small follows by standard arguments, since the semi-linear part of the system is locally Lipschitz continuous: in order to extend the existence result for all time $t > 0$, it suffices to show an a priori L^∞ uniform bound on the solutions.

First of all, we can observe that each equation of the system (2.1) is satisfied by the trivial solution: as a result, the comparison principle applied to each equation implies that the solutions, when defined, are strictly positive for positive t. Using this information, we focus our attention on the equation satisfied by the density u, that is

$$\begin{cases} u_t - D\Delta u = \left(\lambda - \mu u - \sum_{i=1}^N k_i w_i \right) u \\ u(x, 0) = u^0(x). \end{cases} \tag{2.2}$$

Let $U \in C^1(\mathbb{R}^+)$ be the solution of the initial value problem

$$\begin{cases} \dot{U} = \lambda U - \mu U^2 & \text{for } t > 0 \\ U(0) = \max\{\lambda/\mu, \sup_{x \in \Omega} u^0(x)\} > 0. \end{cases}$$

The family of solution U is decreasing in $t > 0$ and $U(t) \to \lambda/\mu$ as $t \to +\infty$: as a result, for any $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ finite such that $U(t) \leq \lambda/\mu + \varepsilon$ for any $t \geq T_\varepsilon$. By a direct inspection, since each w_i is non-negative, we can observe that the solution U is a super-solution to the equation (2.2), and by the comparison principle we have $u(x, t) \leq U(t)$ for all $x \in \Omega$, and it is then bounded uniformly. Taking into account this information, we see that each w_i satisfies

$$\begin{cases} w_{i,t} - d_i \Delta w_i = \left(-\omega_i + k_i u - \mu_i w_i - \beta \sum_{j \neq i} a_{ij} w_j \right) w_i \\ w_i(x, 0) = w_i^0(x). \end{cases} \tag{2.3}$$

Using a similar reasoning as before, we can introduce the auxiliary function $W_i \in \mathcal{C}^1(\mathbb{R}^+)$ solution to the initial value problem

$$\begin{cases} \dot{W}_i = (-\omega_i + k_i U - \mu_i W_i) W_i & \text{for } t > 0 \\ W_i(0) = \sup_{x\in\Omega} w_i^0(x) > 0. \end{cases}$$

Clearly W_i is uniformly bounded in t and moreover, $W_i(t) \to (\lambda k_i - \mu\omega_i)/(\mu\mu_i)$ as $t \to +\infty$: we deduce that for any $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ finite such that $W_i(t) \leq (\lambda k_i - \mu\omega_i)/(\mu\mu_i) + \varepsilon$ for any $t \geq T_\varepsilon$. Moreover, again by direct inspection, since each w_i is non negative and $u \leq U$, see that W_i is a super-solution for (2.3) and thus w_i is bounded uniformly.

The previous uniform upper bounds are enough to ensure that the solution can be extended for all time $t > 0$ and also to conclude the asymptotic estimates. □

Before continuing we recall a result by [CHS78] about the asymptotic behaviour in time of solutions to system of reaction diffusion equations. We let L be the Lipschtiz constant of the semi-linear term in (1.1) on the invariant region of Lemma 2.1, that is, letting

$$F(s_1, \dots, s_N, S) = \begin{pmatrix} \left(-\omega_i + k_i S - \mu_i s_i - \beta \sum_{j\neq i} a_{ij} s_j\right) s_i \\ \left(\lambda - \mu S - \sum_{i=1}^N k_i s_i\right) S \end{pmatrix}$$

we define

$$L = \sup\left\{ |\nabla F(s_1, \dots, s_N, S)| : 0 < s_i < \frac{\lambda k_i - \mu\omega_i}{\mu\mu_i}, 0 < S < \frac{\lambda}{\mu} \right\}.$$

We observe that, thanks to the assumptions, L is finite and positive. We also let

$$d = \min\{d_1, \dots, d_N, D\}$$

and γ_1 as the first non trivial (that is, positive) eigenvalue of the Laplace operator $-\Delta$ in Ω with homogeneous boundary conditions. For any solution $(w_1, \dots, w_N, u)$ of (1.1), we let

$$\bar{w}_i(t) = \frac{1}{|\Omega|}\int_\Omega w_i(x,t)dx, \quad \bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx.$$

Applying [CHS78, Theorem 3.1] to our system (1.1) we have the following result on the asymptotic behavior of the solutions for large time.

Proposition 2.2. *Let*

$$\sigma = d\gamma_1 - L.$$

If $\sigma > 0$, then for any non negative initial condition $(w_1^0, \dots, w_N^0, u^0) \in \mathcal{C}^{0,\alpha}(\Omega)$, the corresponding unique solution of the system (2.1) converges exponential towards spatially homogeneous solutions, that is, for any $0 < \sigma' < \sigma$ there exists $C > 0$ such that

$$\sum_{i=1}^N \|\nabla w_i\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leq Ce^{-\sigma' t}$$

$$\sum_{i=1}^N \|w_i(\cdot,t) - \bar{w}_i(t)\|_{L^\infty(\Omega)} + \|u(\cdot,t) - \bar{u}(t)\|_{L^\infty(\Omega)} \leq Ce^{-\sigma' t/n}.$$

Moreover, the vector $(\bar{w}_1, \dots, \bar{w}_M, \bar{u})$ solves the system of ordinary differential equations

$$\begin{cases} \bar{w}_i' = \left(-\omega_i + k_i\bar{u} - \mu_i\bar{w}_i - \beta\sum_{j\neq i} a_{ij}\bar{w}_j\right)\bar{w}_i + g_i(t) \\ \bar{u}_t' = \left(\lambda - \mu\bar{u} - \sum_{i=1}^N k_i\bar{w}_i\right)\bar{u} + g(t) \end{cases}$$

with

$$\bar{w}_i(0) = |\Omega|^{-1}\int_\Omega w_i(x)^0 dx, \bar{u}(0) = |\Omega|^{-1}\int_\Omega u(x)^0 dx.$$

and

$$\sum_{i=1}^N |g_i(t)| + |g(t)| \leq Ce^{-\sigma' t}.$$

Proof. The proof is a straightforward application of [CHS78, Theorem 3.1]. We only observe that by Lemma 2.1 we know that from any positive initial data and any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that the corresponding unique solution is contained in the region

$$\left\{0 < w_i(x,t) < \frac{\lambda k_i - \mu\omega_i}{\mu\mu_i} + \varepsilon, 0 < u(x,t) < \frac{\lambda}{\mu} + \varepsilon, \forall x \in \Omega\right\}$$

for all $t \geq T_\varepsilon$. Now, if $\sigma > 0$, by regularity of F for any $\varepsilon > 0$ sufficiently small

$$\sigma' = d\gamma_1 - \sup\left\{|\nabla F(s_1, \dots, s_N, S)| : 0 < s_i < \frac{\lambda k_i - \mu\omega_i}{\mu\mu_i} + \varepsilon, 0 < S < \frac{\lambda}{\mu} + \varepsilon\right\} > 0$$

and we can apply [CHS78, Theorem 3.1] to obtain the sought exponential estimates. □

The important consequence of the previous proposition is that the behaviour of the solutions, in the regime $\sigma > 0$ is well described by the corresponding system of ordinary differential equations, and also give us a complete characterisation of the set of stationary solutions of (2.1), which is then given by the (spatially constant) solutions of $F(w_1, \dots, w_N, u) = 0$. For instance, by studying the stability of the stationary solutions (see Proposition 2.4 and Lemma 3.4 below) we will see that in this case, when $\beta > 0$ the only stable ones are those that have $u > 0$ and only one component of $(w_1, \dots, w_N)$ non trivial (and positive). We finally observe that the condition $\sigma > 0$ can be violated in three different ways: (i) lowering the diffusion coefficients, (ii) enlarging the domain or (iii) augmenting the Lipschitz constant L. This last possibility, which is the one that we explore later, can be enforced for instance by taking β large enough.

We now start investigating the equilibria of the system, in particular we want analyse what is the impact of the competition parameter on the possible heterogeneity of the solutions of the system. To do this, we first recall the fundamental result by Dockery et al. [DHMP98] on a related model

$$\begin{cases} w_{i,t} - d_i \Delta w_i = \left(a(x) - \sum_{j=1}^N w_j\right) w_i & \text{in } \Omega \times (0, +\infty) \\ \partial_\nu w_i = 0 & \text{on } \partial\Omega \times (0, +\infty) \end{cases} \tag{2.4}$$

where a is a smooth non constant function such that the principal eigenvalue of each of the elliptic operators

$$\begin{cases} -d_i \Delta w = aw + \lambda w & \text{in } \Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases}$$

denoted by $\lambda(d_i, a)$, is strictly negative (implying, in particular, the instability of the trivial solution). Exploiting the particular symmetric structure of the interaction/competition term, Dockery et al. were able to show that the only asymptotically and hyperbolic stable equilibrium of the system is the stationary solution that has all the components w_i trivial except for the one with the smallest diffusion coefficient d_i. Moreover, the same result holds if we introduce a small mutation term in the system, which in terms imply also an evolutionary advantage for small diffusion rates. The classic interpretation of this result is that, since the densities w_i in (2.4) are equivalent if not for the diffusion rates, the density which can concentrate more on favorable zones will benefit.

In what follows, we shall show that this is not the case for the model we are considering, and in particular we shall prove that for β sufficiently large, all the solutions that have only one nontrivial density of predators are asymptotically and hyperbolic stable.

Remark 2.3. In order to justify the link between the model (2.4) and our model (2.1), let us consider the limit case of (2.1) in which the density u has a very fast dynamic with respect to the other components, that is, let us assume that for each $t > 0$, the density u reaches instantaneously its non-trivial inviscid equilibrium state,

$$\lambda u - \mu u^2 - u \sum_{i=1}^N k_i w_i = 0 \implies u = \frac{1}{\mu}\left(\lambda - \sum_{i=1}^N k_i w_i\right).$$

Substituting the previous identity in the equations satisfied by w_i we obtain

$$w_{i,t} - d_i \Delta w_i = \left(\frac{k_i \lambda}{\mu} - \omega_i - \frac{k_i}{\mu} w_i - \sum_{j \neq i}\left(\beta a_{ij} + \frac{k_i}{\mu}\right) w_j\right) w_i$$

In the simplified case $k_i = \mu$, $\omega_i = \omega$ and $\beta = 0$, we obtain finally

$$w_{i,t} - d_i \Delta w_i = \left(\lambda - \omega - \sum_{j=1}^{N} w_j \right) w_i = \left(a - \sum_{j=1}^{N} w_j \right) w_i$$

where a is, in this oversimplified case, a constant: later we shall consider a generalization of the model that avoids this inconvenience. More details can be found in [BZa].

We have the following

Proposition 2.4. *For a fixed $i \in \{1, \dots, N\}$, let W be the stationary solution of* (1.1) *which as only the i-th densities of predator which is nontrivial, than W is constant and $W = (0, \dots, \tilde{w}_i, \dots, 0, \tilde{u})$ with*

$$\tilde{w}_i = \frac{\lambda k_i - \mu \omega_i}{k_i^2}, \quad \tilde{u} = \frac{\omega_i}{k_i}.$$

There exists $\bar{\beta} \geq 0$ such that if $\beta \geq \bar{\beta}$ then W is asymptotically and hyperbolic stable with respect to perturbations in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$. More explicitly, $\bar{\beta}$ must satisfy the system of inequalities

$$\bar{\beta} \geq \frac{k_j}{a_{ji} \tilde{w}_i} \left(\frac{\omega_i}{k_i} - \frac{\omega_j}{k_j} \right) \qquad \forall j \neq i.$$

Proof. First of all, by [Mim79, Theorem 1] and [CS77], we have that the only solution of the system with only the i-th density of predator non trivial is the constant solution W. The study of the stability of this solution is based on a simple analysis of the linearized system around it: we consider the Fréchet differential around W of the operator describing the system, which is given by

$$L(W)[w_1, \dots, w_N, u] = \begin{cases} -d_i \Delta w_i - k_i \tilde{w}_i u + \beta \tilde{w}_i \sum_{j \neq i} a_{ij} w_j \\ -d_j \Delta w_j + \left[k_j \left(\frac{\omega_j}{k_j} - \frac{\omega_i}{k_i} \right) + \beta \tilde{w}_i a_{ji} \right] w_j \quad \text{for } j \neq i \\ -D \Delta u + \mu \frac{\omega_i}{k_i} u + \omega_i w_i \end{cases}$$

for all $(w_1, \dots, w_N, u) \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ with homogenous Neumann boundary conditions. To ensure the stability of the solution we need to show that the spectrum of L is contained in $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$, that is for any $(w_1, \dots, w_N, u) \neq 0$ and $\gamma \in \mathbb{C}$

$$L(W)[w_1, \dots, w_N, u] = \gamma(w_1, \dots, w_N, u) \iff \Re(\gamma) > 0.$$

In the previous system, the components corresponding to $j \neq i$ are decoupled from the others, and thus their presence does not compromise the stability of W if and only if

$$k_j \left(\frac{\omega_j}{k_j} - \frac{\omega_i}{k_i} \right) + \beta \tilde{w}_i a_{ji} > 0 \qquad \forall j \neq i$$

which gives the condition established by the proposition; indeed, under this assumption the components w_j with $j \neq i$ are necessarily trivial. Let us show that this condition is enough to ensure the stability: we suppose that the previous system of inequalities holds but there exist $(w_1, \dots, w_N, u) \neq 0$ and $\gamma \in \mathbb{C}$ with $\Re(\gamma) \leq 0$ solution to

$$L(W)[w_1, \dots, w_N, u] = \gamma(w_1, \dots, w_N, u).$$

Then necessarily $w_j = 0$ for all $j \neq i$, and the system is reduced to

$$\begin{cases} -d_i \Delta w_i = \gamma w_i + k_i \tilde{w}_i u \\ -D \Delta u = -\omega_i w_i + \left(\gamma - \mu \frac{\omega_i}{k_i} \right) u \\ \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

Since any weak solution to the previous system is regular, the stability in $\mathcal{C}^{2,\alpha}(\Omega)$ can be deduced from the solvability of the system in $H^1(\Omega)$. To analyse it, let $\{(\gamma_h, \psi_h)\}_{h \in \mathbb{N}}$ the spectral resolution of the Laplace operator with homogeneous Neumann boundary condition in Ω (let us recall that $\gamma_0 = 0$ and $\gamma_h > 0$ for $h > 0$); since $\{\psi_h\}_{h\in\mathbb{N}}$ is a complete base of $L^2(\Omega)$, we can write

$$w_i = \sum_{h=0}^{\infty} a_h \psi_h \quad \text{and } u = \sum_{h=0}^{\infty} b_h \psi_h$$

as series converging in $L^2(\Omega)$. Inserting these relations in (2.5) and using the orthogonality of the eigenfunctions, we see that it is equivalent to the sequence of algebraic eigenvalue problems

$$\begin{cases} d_i \gamma_h a_h - k_i \tilde{w}_i b_h = \gamma a_h \\ \left(D\gamma_h + \mu \frac{\omega_i}{k_i}\right) b_h + \omega_i a_h = \gamma b_h \end{cases} \quad \text{for } h \in \mathbb{N}.$$

By direct inspection, we can observe that $a_h = 0$ if and only if $b_h = 0$. Thus, solving the first equation in b_h and substituting the result in the second, we find that γ must be a solution to

$$\left(D\gamma_h + \mu \frac{\omega_i}{k_i} - \gamma\right)(d_i\gamma_h - \gamma) + k_i \tilde{w}_i \omega_i = 0$$

that is

$$\gamma = \frac{1}{2}\left[\left((D + d_i)\gamma_h + \mu\frac{\omega_i}{k_i}\right) \pm \sqrt{\left((D + d_i)\gamma_h + \mu\frac{\omega_i}{k_i}\right)^2 - 4k_i\tilde{w}_i\omega_i}\right]$$

and in particular $\Re(\gamma) > 0$. □

Let us observe that, the diffusion rates do not play any role in the stability of the solutions, while a crucial quantity is given by the ratio ω_i/k_i. In particular if i is such that

$$\frac{\omega_i}{k_i} < \frac{\omega_j}{k_j} \qquad \forall j \neq i$$

that the solution W is asymptotically stable also in a slight cooperative environment, that is for $\beta < 0$ small. This is a consequence of the fact that the semi-trivial solutions are constant and the different diffusion rates do not play a direct role in the stability of the solution (i.e., advantage of low/high diffusion rate). In this setting, the quantity ω_i/k_i can be interpreted as the fitness of the i-th population.

One could then wonder whether the previous stability result is a spurious consequence of the fact the simple solutions are constant or of another specific features of this particular formulation of the system: in order to confute this doubt, we shall now adapt the proof to a very general framework. Let us consider the following operator

$$\mathcal{S}_\beta(\mathbf{v}) := \begin{cases} \mathcal{L}_i w_i - \left[f_i(x, u, w_i) - \beta \sum_{j\neq i} g_{ij}(x, w_i, w_j)\right] w_i & \text{for all } i \in \{1, \dots, N\} \\ \mathcal{L}u - f(x, u, w_1, \dots, w_N)u \end{cases}$$

defined for $\mathbf{v} = (w_1, \dots, w_N, u)$ in the set

$$X(\Omega) = \left\{\mathbf{v} \in \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^{N+1}) : \partial_\nu^{\mathcal{L}_i} w_i = \partial_\nu^{\mathcal{L}} v = 0 \text{ on } \partial\Omega\right\}$$

where each $\mathcal{L}_i$ and $\mathcal{L}$ stands for a linear (strongly) elliptic operator of the form

$$\mathcal{L}_i w_i = -\operatorname{div}(A_i(x)\nabla w_i), \quad \mathcal{L}u = -\operatorname{div}(A(x)\nabla u)$$

for some smooth and uniformly elliptic symmetric matrices A_i and A. We assume in the following that all the terms in the operator $\mathcal{S}_\beta$ are smooth enough to justify the following computations, and moreover we suppose that there exists positive constants C such that for any $\mathbf{v} \in X(\Omega)$ of non negative components we have

$$\begin{cases} f_i(x, u, w_i) \leq C(1 + u - w_i) \\ f(x, u, w_1, \dots, w_N) \leq C(1 - u) \\ g_{ij}(x, w_i, w_j) \geq 0 \end{cases}$$

Based on the previous notation, a function $\mathbf{v} \in X(\Omega)$ is a solution of the generalized model if

$$\mathcal{S}_\beta(\mathbf{v}) = 0$$

while a function $\mathbf{v} \in \mathcal{C}^1(\mathbb{R}^+; X(\Omega)) \cap \mathcal{C}(\overline{\mathbb{R}^+}; X(\Omega))$ is a solution to the parabolic model if

$$\begin{cases} \partial_t \mathbf{v} + \mathcal{S}_\beta(\mathbf{v}) = 0 & t > 0 \\ \mathbf{v}(0) = \mathbf{v}_0 & \mathbf{v}_0 \in X(\Omega). \end{cases}$$

Using the previous assumptions, we have

Lemma 2.5. *For any non-negative initial datum $\mathbf{v}_0 \in X(\Omega)$ there exists a unique solution $\mathbf{v}$ of the previous parabolic problem. Moreover, there exists $T > 0$ and $M > 0$, independent of β, such that*

$$0 \le w_1(t,x), \dots, w_N(t,x), u(t,x) \le M \qquad \text{for all } t \ge T,\ x \in \overline{\Omega}.$$

If there exist $i \in \{1, \dots, N\}$, $t > 0$ and $x_0 \in \overline{\Omega}$ such that $w_i(t, x_0) = 0$ (respectively, $u(t, x_0) = 0$), than $w_i \equiv 0$ (respectively, $u \equiv 0$).

Proof. The proof follows directly from the maximum principle, and thus it is omitted. For comparison, we also recall Lemma 2.1. □

In complete analogy, we have a corresponding result for the stationary model. Among the class of all possible solutions, we are interested in the particular case of solutions that have only one component among the first N which is non-trivial.

Definition 2.6. For a given $i \in \{1, \dots, N\}$, a solution $\mathbf{v} \in X(\Omega)$ is said to be i-simple if

$$w_i \neq 0, u \neq 0 \quad \text{ while } w_j \equiv 0 \text{ for all } j \neq i.$$

Let us observe that if $\mathbf{v} \in X(\Omega)$ is an i-simple solution for $\mathcal{S}_\beta$, then it is i-simple solution for any β.

For a given solution $\mathbf{v} \in X(\Omega)$, let $L(\mathbf{v})$ be the Fréchet derivates of $\mathcal{S}_\beta$ in $X(\Omega)$, that is for any $\boldsymbol{\varphi} \in X(\Omega)$ we have

$$\begin{aligned} L(\mathbf{v})[\boldsymbol{\varphi}] &= \lim_{\varepsilon \to 0} \frac{\mathcal{S}_\beta(\mathbf{v} + \varepsilon\boldsymbol{\varphi}) - \mathcal{S}_\beta(\mathbf{v})}{\varepsilon} \\ &= \begin{cases} \mathcal{L}_i\varphi_i - \left[f_i(x,u,w_i) - \beta \sum_{j\neq i} g_{ij}(x,w_i,w_j)\right]\varphi_i \\ \qquad -f_{i,u}(x,u,w_i)w_i\varphi - f_{i,w_i}(x,u,w_i)w_i\varphi_i \\ \qquad +\beta\sum_{j\neq i} g_{ij,w_i}(x,w_i,w_j)w_i\varphi_i + \beta\sum_{j\neq i} g_{ij,w_j}(x,w_i,w_j)w_i\varphi_j \\ \mathcal{L}\varphi - f(x,u,w_1,\dots,w_N)\varphi - f_{,u}(x,u,w_1,\dots,w_N)\varphi \\ \qquad -\sum_{i=1}^N f_{,w_i}(x,u,w_1,\dots,w_N)\varphi_i \end{cases} \end{aligned}$$

Analogously, for any fixed $i \in \{1, \dots, N\}$ we defined the i-th partial derivates $L_i(\mathbf{v})$ as the Fréchet derivates of $\mathcal{S}_\beta$ in $X(\Omega)$ with respect to the direction $\boldsymbol{\varphi} \in X(\Omega)$ such that $\boldsymbol{\varphi} = (0, \dots, \varphi_i, \dots, 0, \varphi)$, that is

$$L_i(\mathbf{v})[\boldsymbol{\varphi}] = \begin{cases} \mathcal{L}_i\varphi_i - \left[f_i(x,u,w_i) - \beta \sum_{j\neq i} g_{ij}(x,w_i,w_j)\right]\varphi_i \\ \qquad -f_{i,u}(x,u,w_i)w_i\varphi - f_{i,w_i}(x,u,w_i)w_i\varphi_i \\ \qquad +\beta\sum_{j\neq i} g_{ij,w_i}(x,w_i,w_j)w_i\varphi_i \\ 0 \qquad \text{for } j \neq i \\ \mathcal{L}\varphi - f(x,u,w_1,\dots,w_N)\varphi - f_{,u}(x,u,w_1,\dots,w_N)\varphi \\ \qquad -f_{,w_i}(x,u,w_1,\dots,w_N)\varphi_i \end{cases}$$

Accordingly, we have recall that a solution $\mathbf{v} \in X(\Omega)$ is stable if any non-trivial solution $(\gamma, \boldsymbol{\varphi})$ of

$$L(\mathbf{v})[\boldsymbol{\varphi}] = \gamma\boldsymbol{\varphi}$$

has necessarily $\Re(\gamma) > 0$. For i-simple solutions we have

Definition 2.7. For a given $i \in \{1, \dots, N\}$, an i-simple solution $\mathbf{v} \in X(\Omega)$ is *internally* stable if any non-trivial solution $(\gamma, \boldsymbol{\varphi})$ of

$$L_i(\mathbf{v})[\boldsymbol{\varphi}] = \gamma\boldsymbol{\varphi}$$

with $\boldsymbol{\varphi} = (0, \dots, \varphi_i, \dots, 0, \varphi)$ we have necessarily $\Re(\gamma) > 0$.

Clearly, if an i-simple solution is stable it is also internally stable: under suitable conditions, the inverse is true.

Proposition 2.8. *For a given* $i \in \{1, \dots, N\}$, *let us assume that*

$$\inf_{x\in\Omega} g_{ji}(x, 0, s) > 0 \qquad \text{for all } s > 0 \text{ and } j \neq i.$$

If $v \in X(\Omega)$ *be an* i*-simple internally stable solution* $\mathbf{v} \in X(\Omega)$, *then there exists* $\bar{\beta}$ *such that* $\mathbf{v}$ *is a stable solution for all* $\beta > \bar{\beta}$.

Proof. The i-simple solution $\mathbf{v} = (0, \dots, w_i, \dots, 0, u)$ is stable if

$$\begin{cases} \mathcal{L}_i\varphi_i - \left[f_i(x,u,w_i) - \beta \sum_{j\neq i} g_{ij}(x,w_i,0)\right]\varphi_i \\ \qquad -f_{i,u}(x,u,w_i)w_i\varphi - f_{i,w_i}(x,u,w_i)w_i\varphi_i \\ \qquad +\beta\sum_{j\neq i} g_{ij,w_i}(x,w_i,0)w_i\varphi_i + \beta\sum_{j\neq i} g_{ij,w_j}(x,w_i,0)w_i\varphi_j = \lambda\varphi_i \\ \mathcal{L}_j\varphi_j - \left[f_j(x,u,0) - \beta g_{ji}(x,0,w_i)\right]\varphi_j = \lambda\varphi_j \\ \mathcal{L}\varphi - f(x,u,0,\dots,w_i,\dots,0)\varphi - f_{,u}(x,u,0,\dots,w_i,\dots,0)\varphi \\ \qquad -\sum_{i=1}^N f_{,w_i}(x,u,0,\dots,w_i,\dots,0)\varphi_i = \lambda\varphi \end{cases}$$

has a nontrivial solution $\boldsymbol{\varphi} \in X(\Omega)$ if and only if $\Re(\lambda) > 0$. Let us consider the equations of index $j \neq i$, which are decoupled from the other equations in the system, we have

$$\begin{cases} \mathcal{L}_j\varphi_j = \left[f_j(x,u,0) - \beta g_{jh}(x,0,w_i) + \lambda\right]\varphi_j & \text{in } \Omega \\ \partial_\nu^{\mathcal{L}_j}\varphi_j = 0 & \text{on } \partial\Omega \end{cases}$$

As the operator $\mathcal{L}_j$ is self-adjoint[1], any non-trivial solution of the system must have $\lambda \in \mathbb{R}$. Since the solution $\mathbf{v}$ is i-simple solution, by the maximum principle we have that

$$\inf_{x\in\Omega} w_i(x) = c > 0.$$

As a result, thanks to our assumptions, there exists $\bar{\beta} \geq 0$ such that

$$\bar{\beta} \geq \sup_{x\in\Omega} \frac{f_j(x,u,0)}{g_{ji}(x,0,w_i)} \qquad \text{for all } j \neq i.$$

Choosing $\beta > \bar{\beta}$ and testing the equation in φ_j by φ_j itself, we obtain

$$\int_\Omega A_j(x)\nabla w_j \cdot \nabla w_j = \int_\Omega \left[f_j(x,u,0) - \beta g_{ji}(x,0,w_i) + \lambda\right]\varphi_j^2 < \lambda \int_\Omega \varphi_j^2$$

thus either $\lambda > 0$ or the component $\varphi_j = 0$. On the other hand, assuming that $\Re(\lambda) \leq 0$, we find a contradiction with the internal stability of the solution $\mathbf{v}$. ☐

3. Stationary model: bifurcation analysis

We continue the investigation of the asymptotic properties of the system (1.1), this time by studying the set of solutions of the corresponding elliptic problem.We consider here the model (1.1) under the assumption that the domain Ω is occupied by only two indistinguishable groups of predator, that is

$$\begin{cases} -d\Delta w_1 = (-\omega + ku - \beta w_2)\, w_1 & \text{in } \Omega \\ -d\Delta w_2 = (-\omega + ku - \beta w_1)\, w_2 & \text{in } \Omega \\ -D\Delta u = (\lambda - \mu u - k(w_1 + w_2))\, u & \text{in } \Omega \\ \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

for which we look for solutions $(w_1, w_2, u) \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. Let us point out that we let $\mu_1 = \mu_2 = 0$, but the results that will we shown in the following can be easily generalized to the case of positive saturation

[1] More precisely, the operator is self-adjoint if seen as an operator acting on $H^1(\Omega)$ functions, and the conclusion can be reached using the regularity assumptions on its coefficients.

coefficients (though the computations are inevitably harder). Since we are looking for stationary solutions, the system can be simplified by some linear substitutions: indeed, if we let

$$u \mapsto \frac{d}{D}u, \ \lambda \mapsto \lambda D, \ \mu \mapsto \mu\frac{D^2}{d}, \ k \mapsto kD, \ \omega \mapsto \omega d, \ \beta \mapsto \beta d$$

we can reformulate the system as

$$\begin{cases} -\Delta w_1 = (-\omega + ku - \beta w_2)\, w_1 & \text{in } \Omega \\ -\Delta w_2 = (-\omega + ku - \beta w_1)\, w_2 & \text{in } \Omega \\ -\Delta u = (\lambda - \mu u - k(w_1 + w_2))\, u & \text{in } \Omega \\ \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

We recall the definition of the set

$$X(\Omega) := \left\{(w_1, w_2, u) \in \mathcal{C}^{2,\alpha}(\overline{\Omega}) : \partial_\nu w_1 = \partial_\nu w_2 = \partial_\nu u = 0 \text{ on } \partial\Omega\right\}.$$

We are interested in non negative solutions of the system. Letting all the other parameters of the model fixed, we shall study the set of the solutions of (3.1) by varying the competition strength β: for this reason, in the following, we will need estimate that are uniform in β, similarly to what we did for the parabolic model (1.1). Let us recall that the assumption (H) holds, that is $\lambda k > \mu\omega$.

We start by recalling a result concerning the regularity of the solution of the system (3.1). This result is nothing but the elliptic counterpart of [BZb], to which we also refer the interested reader.

Proposition 3.1. *Let $(w_1, w_2, u) \in H^1(\Omega)$ be a non negative weak solution to (3.1). Then*

- *the solutions are classical and, more precisely, we have that $(w_1, w_2, u) \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^{2,\alpha}(\overline{\Omega})$ for any $\alpha < 1$ and the regularity is bounded only by the regularity of Ω;*
- *(w_1, w_2, u) are non negative and bounded uniformly in β, that is*

$$\begin{cases} w_1 \geq 0, w_2 \geq 0, 0 \leq u \leq \lambda/\mu \\ u + w_1 + w_2 \leq \dfrac{(\lambda+\omega)^2}{4\mu\omega} \end{cases}$$

  and either all the inequalities are strict or the solution is constant;
- *there exists a constant $C > 0$ (independent of β) such that*

$$\|(w_1, w_2)\|_{Lip(\overline{\Omega})} + \|(w_1 - w_2, u)\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq C$$

- *if $(w_{1,\beta}, w_{2,\beta}, u_\beta)$ is any family of solution to (3.1) that satisfy the assumptions, then up to a subsequence, there exists $(w_1, w_2) \in Lip(\overline{\Omega})$ and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ for any $\alpha < 1$ such that $w_1 w_2 = 0$ in Ω,*

$$(w_{1,\beta}, w_{2,\beta}) \to (w_1, w_2) \text{ in } \mathcal{C}^{0,\alpha} \cap H^1(\overline{\Omega}), u_\beta \to u \text{ in } \mathcal{C}^{2,\alpha}(\overline{\Omega}),$$

  for any $\alpha < 1$, and moreover (w_1, w_2, u) are solutions to

$$\begin{cases} -\Delta(w_1 - w_2) = -\omega(w_1 - w_2) + k(w_1 - w_2)u & \text{in } \Omega \\ -\Delta u = \lambda u - \mu u^2 - k(w_1 + w_2)u & \text{in } \Omega \\ \partial_\nu(w_1 - w_2) = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

  In particular $\{x \in \Omega : w_1(x) = w_2(x)\}$ is a rectifiable set of codimension 1, made of the union of a finite number of $\mathcal{C}^{1,\alpha}$ smooth sub-manifolds.

Proof. Considering the equation satisfied by u, we have

$$-\Delta u = \lambda u - \mu u^2 - k(w_1 + w_2)u \leq \lambda u - \mu u^2$$

and the left hand side is negative if $u > \lambda/\mu$. Moreover, letting $v = w_1 + w_2 + u$ and summing the three equations in the system, we obtain

$$-\Delta v = -\omega(w_1 + w_2) - 2\beta w_1 w_2 + \lambda u - \mu u^2 \leq -\omega v + (\lambda + \omega)u - \mu u^2 \leq -\omega v + \frac{(\lambda+\omega)^2}{4\mu}$$

and the left hand side is again negative if $v > (\lambda + \omega)^2/(4\mu\omega)$. The thesis thus follows by the classical maximum principle.

Once the L^∞ uniform estimate is settled, we can continue by decoupling the equations. In particular, considering the equation satisfied by the density u (which depends on β only through the uniformly bounded (w_1, w_2)) we have

$$\begin{cases} -\Delta u = \lambda u - \mu u^2 - k(w_1 + w_2)u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

which implies, by the standard elliptic estimates, that there exists $C > 0$ independent of β, such that

$$\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C \qquad \text{for all } \alpha \in (0,1).$$

Passing to the equations satisfied by (w_1, w_2), we have

$$\begin{cases} -\Delta w_1 = (-\omega + ku)w_1 - \beta w_1 w_2 & \text{in } \Omega \\ -\Delta w_2 = (-\omega + ku)w_2 - \beta w_1 w_2 & \text{in } \Omega \\ \partial_\nu w_1 = \partial_\nu w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to the results in [CTV05a] (see also [SZ15] for a different proof that can covers also the case of more than two compenents)[2] we deduce that there exists yet another constant $C > 0$ such that

$$\|(w_1, w_2)\|_{Lip(\overline{\Omega})} \leq C \qquad \text{independently of } \beta.$$

It then suffices to use these new and stronger uniform estimates in the equation satisfied by u in order to obtain the full uniform estimate

$$\|(w_1, w_2)\|_{Lip(\overline{\Omega})} + \|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq C \qquad \text{independently of } \beta.$$

The concluding assertions of the proposition follow by the Ascoli-Arzela compactness criterion and the specific structure of the equation satisfied by $w_1 - w_2$. □

While Proposition 3.1 gives a precise description of the solutions of the system (3.1), it contains no information about the existence of such solutions. In the following, our aim is to complete this gap, showing that the set of solutions is rich. Before doing so, we need to introduce some notation.

For a given solution $(w_1, w_2, u) \in X(\Omega)$ of the system (3.1), the Fréchet derivate in $X(\Omega)$ associated to (3.1) computed at (w_1, w_2, u) is given by

$$L_\beta \boldsymbol{\varphi} = -\Delta \boldsymbol{\varphi} - A_\beta \boldsymbol{\varphi}, \quad \text{for any } \boldsymbol{\varphi} \in X(\Omega)$$

where $A_\beta = A_\beta(w_1, w_2, u) \in \mathcal{C}^{2,\alpha}(\overline{\Omega}, \mathbb{R}^{3\times 3})$ is

$$A = A_\beta = \begin{pmatrix} -\omega + ku - \beta w_2 & -\beta w_1 & k w_1 \\ -\beta w_2 & -\omega + ku - \beta w_1 & k w_2 \\ -ku & -ku & \lambda - 2\mu u - k w_1 - k w_2 \end{pmatrix}.$$

The solution (w_1, w_2, u) is said to be (strongly linearly) stable if any non-trivial solution $(\gamma, \boldsymbol{\varphi})$ of the linearized equation

$$L_\beta \boldsymbol{\varphi} = \gamma \boldsymbol{\varphi}$$

has necessarily $\Re(\gamma) > 0$; it is said (strongly linearly) unstable if on the contrary there exist a non-trivial solution with $\Re(\gamma) < 0$.

Remark 3.2. Let us observe that if the solution (w_1, w_2, u) in the previous definition is constant, its stability can be directly deduced by the spectrum of the matrix A_β.

We now proceed with the study of the constant solutions of the system (3.1). We start with the simplest scenario, that is the limit case $\beta = 0$. Under this assumption, since the densities of predators do not interact directly with each other, we can simplify drastically the system and give a complete description of the set of solutions of the system.

[2] It should be pointed out that the cited papers deal with uniform estimates for the solutions inside of the domain Ω, that is to say in any compact subset of Ω, and not at the boundary. But since the boundary conditions are of homogeneous Neumann type, it suffices to flatten locally the boundary and reflect the solutions across it in order to obtain the desired uniform estimate in $\overline{\Omega}$.

Lemma 3.3. *Let $\beta = 0$, then the unique non negative solutions (w_1, w_2, u) to the system* (3.1) *are the two unstable constant solutions*

$$(0,0,0), \left(0,0,\frac{\lambda}{\mu}\right)$$

and the one-parameter family of (weakly) stable ones

$$s \in [0,1] \mapsto \left(\frac{\lambda k - \mu\omega}{k^2}s, \frac{\lambda k - \mu\omega}{k^2}(1-s), \frac{\omega}{k}\right).$$

Proof. In this proof, we shall only classify the solutions; the study of the stability will be postponed in Lemma 3.4, where we shall address more generally the question about stability of constant solutions for $\beta \geq 0$.

Since for $\beta = 0$ the densities of predators do not interact directly with each other, we can simplify the system introducing the new variable $V = w_1 + w_2$, which, together with u is a solution of the classical Lotka-Volterra system

$$\begin{cases} -\Delta V = -\omega V + kVu & \text{in } \Omega \\ -\Delta u = \lambda u - \mu u^2 - kVu & \text{in } \Omega \\ \partial_\nu V = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

From the results in [Mim79, Theorem 1] we have that the previous system has only constant solutions, that is solution to the algebraic system

$$\begin{cases} (ku - \omega)V = 0 \\ (\lambda - \mu u - kV)u = 0 \end{cases}$$

If $V = 0$, we have the solutions $u = 0$ or $u = \lambda/\mu$ which correspond to the first two solutions in the thesis (recall that w_1 and w_2 are non negative, that is, in this case, $w_1 = w_2 = 0$). On the other hand, if $u = \omega/k$, we obtain the solution $V = w_1 + w_2 = (\lambda k - \mu\omega)/k^2$. Substituting this information in (3.1) we obtain that both w_1 and w_2 are harmonic functions, hence constants. □

As we shall see later, the value $\beta = 0$ corresponds to a bifurcation point of multiplicity one for the system (3.1) around the solution

$$(w_1, w_2, u) = \left(\frac{\lambda k - \mu\omega}{2k^2}, \frac{\lambda k - \mu\omega}{2k^2}, \frac{\omega}{k}\right),$$

so that the one-parameter family of solutions of Lemma 3.3 is nothing but the branch of solutions generating from it.

Lemma 3.4. *The system* (3.1) *admits four different types of constant solutions:*

(*a*) *the trivial solution* $(0,0,0)$*, which is strongly unstable;*

(*b*) *the semi-trivial solutions*

$$w_1 = 0, w_2 = 0, u = \frac{\lambda}{k}$$

which is strongly unstable;

(*c*) *the semi-trivial solutions*

$$w_1 = \frac{\lambda k - \mu\omega}{k^2}, w_2 = 0, u = \frac{\omega}{k} \quad \text{and } w_1 = 0, w_2 = \frac{\lambda k - \mu\omega}{k^2}, u = \frac{\omega}{k}$$

which are strongly stable;

(*d*) *the family of non trivial solutions*

$$w_1 = w_2 = \frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}, u = \frac{\lambda\beta + 2k\omega}{\mu\beta + 2k^2}$$

which are unstable for $\beta > 0$. In particular, in this latter case,

$$\sigma(A_\beta) = \left\{\beta\frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}, \gamma_{1,\beta}, \gamma_{2,\beta}\right\}$$

where $\gamma_{1,\beta}$ and $\gamma_{2,\beta}$ are two, possibly complex conjugate, eigenvalues with negative real part.

Proof. The proof is a rather straightforward computation, but we will report it in order to give also interpretation on the results.

The trivial solution $(0,0,0)$ corresponds to the matrix

$$A_\beta = \begin{pmatrix} -\omega & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

which is already in a diagonal form. The instability is caused by the eigenvalues $\lambda > 0$, which corresponds to the constant eigenfunction $(0,0,1)$. As a result, in complete accordance with other biological models, it implies that a logistic growth law in the prey population is responsible for an exponential growth, uniform in all the domain Ω, at least when the population is small. Let us observe that none of the spectral and stability properties of the trivial solution depends on the competition β.

Similar computations hold for the semi-trivial solution $(0,0,\lambda/\mu)$, whose matrix is

$$A_\beta = \begin{pmatrix} \frac{\lambda k - \mu\omega}{\mu} & 0 & 0 \\ 0 & \frac{\lambda k - \mu\omega}{\mu} & 0 \\ -\lambda k/\mu & -\lambda k/\mu & -\lambda \end{pmatrix}.$$

The semi-trivial solutions are, due to the symmetry of the system, completely analogous. Let us focus for example on the solution $w_1 = (\lambda k - \mu\omega)/k^2$, $w_2 = 0$ and $u = \omega/k$. In this case the matrix A_β becomes

$$A_\beta = \begin{pmatrix} 0 & -\beta w_1 & k w_1 \\ 0 & -\beta w_1 & 0 \\ -\omega & -\omega & -\mu\omega/k \end{pmatrix}.$$

A direct computation shows that the spectrum of A_β is given by

$$\gamma = -\beta w_1, -\frac{\mu\omega/k \pm \sqrt{(\mu\omega/k)^2 - 4k\omega w_1}}{2},$$

implying strong stability of the semi-trivial solutions. As already observed in the previous section, this result in unchanged even for the model with different parameters for the two populations of predators, as long as $\beta > 0$.

In the case of non trivial constant solutions, recalling that $w_1 = w_2$, the matrix A_β reduces to

$$A_\beta = \begin{pmatrix} 0 & -\beta w_1 & k w_1 \\ -\beta w_1 & 0 & k w_1 \\ -ku & -ku & -\mu u \end{pmatrix}.$$

By direct inspection (comparing the first two row of A_β), we see that $\beta w_1 \geq 0$ is an eigenvalue, implying in particular that the nontrivial solutions are unstable for $\beta > 0$. Using this information, we can factorize the characteristic polynomial of A_β, yielding to

$$\det(A - \gamma \mathrm{Id}) = (\gamma - \beta w_1)\left[\gamma^2 + (\beta w_1 + \mu u)\gamma + (2k^2 u w_1 + \beta\mu u w_1)\right] = 0$$

that is

$$\gamma = \beta w_1, -\frac{(\beta w_1 + \mu u) \pm \sqrt{(\beta w_1 + \mu u)^2 - (2k^2 u w_1 + \beta\mu u w_1)}}{2},$$

and this concludes the proof. □

The set of non-trivial constant solutions undergoes a transformation when $\beta = 0$, see Lemma 3.3. Moreover, the spectrum of the matrix A_0, computed on the linear set of solutions is

$$\gamma = 0, -\frac{\mu\omega/k \pm \sqrt{(\mu\omega/k)^2 - 4k\omega(w_1 + w_2)}}{2}.$$

The trivial eigenvalue underlines the degeneracy of the constant solutions, as they form a linear subspace, while the other two strictly negative eigenvalues confirm that this set of solutions is stable with respect to perturbations that move apart from this configuration, i.e. non homogeneous perturbation (see Proposition 2.2).

Remark 3.5. The stability of the solutions belonging to the classes (a), (b) and (c) does not depend on β. More precisely, in the classes (a) and (b) the spectrum of A_β is independent of β, while in the third case (c) the spectrum is also contained in $\mathbb{C}^- := \{z \in \mathbb{C} : \Re(z) < 0\}$.

We can say more about constant solution, and in particular we have that if the component u is constant, so are the other components.

Lemma 3.6. *For a solution (w_1, w_2, u) of (3.1), if u is constant, so are the other components.*

Proof. The case for β is already object of Lemma 3.3, thus we can assume $\beta > 0$. Starting from the equation in u, assuming u a positive constant, we find that necessarily

$$w_1 + w_2 = \frac{\lambda}{k}.$$

Substituting the previous identity in the equation for w_i, $i = 1, 2$, we obtain

$$\begin{cases} -\Delta w_i = \left(-\omega + ku - \beta\frac{\lambda}{k} + \beta w_i\right) w_i \\ \partial_\nu w_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Summing up the equation, we obtain moreover

$$w_1^2 + w_2^2 = \frac{\lambda}{k}\left(\frac{\lambda}{k} - \frac{ku - \omega}{\beta}\right) > 0$$

As a result, we have obtained the identities

$$w_1 + w_2 = a, \qquad w_1^2 + w_2^2 = b$$

for some positive constant a and b: we claim that these imply that both w_1 and w_2 are constant. Indeed taking the gradient of both expression we see that

$$(w_1 - w_2)\nabla w_1 = (w_1 - w_2)\nabla w_2 = 0$$

and the claim follows. Using this information, it is also possible to compute explicitly the solutions, and in particular we find $u = \omega/k$. ☐

We are mostly interested in the case of completely non-trivial solutions, which is investigated further in the following lemma. Let $\mathscr{S}_\beta$ stand for the set of completely non-trivial constant solutions (w_1, w_2, u) of the form

$$w_1 = w_2 = \frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}, u = \frac{\lambda\beta + 2k\omega}{\mu\beta + 2k^2}.$$

Lemma 3.7. *Let $(w_1, w_2, u) \in \mathscr{S}_\beta$. The eigenvalues of A_β behave as*

$$\beta\frac{\lambda k - \mu\omega}{\mu\beta + 2k^2} \sim \frac{\lambda k - \mu\omega}{\mu}, \quad \gamma_{1,\beta} \sim -\left(\lambda + \frac{\lambda k - \mu\omega}{\mu}\right), \quad \gamma_{2,\beta} \to 0^- \qquad \text{as } \beta \to \infty.$$

In particular, the supremum of the spectrum of the matrix A_β is described, in terms of β, by the curve

$$\beta \mapsto \beta\frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}.$$

Moreover, the supremum of the spectrum is monotone increasing in β and its limit as $\beta \to \infty$ can be made as large as wanted by taking μ small accordingly. In particular, in the limit case $\mu = 0$ the spectrum is unbounded.

The unstable direction of A_β is spanned by the eigenvector $(1, -1, 0)$.

We can now show a result concerning the existence of non constant solutions: our construction is implicit and uses the topological degree argument through a bifurcation analysis of the set of constant solutions. Let $0 = \gamma_0 < \gamma_1 \le \gamma_2 \le \dots$ be the sequence of eigenvalues of the Laplace operator with homogeneous Neumann boundary conditions and let $\{\psi_i\}$ be the corresponding eigenfunctions

$$\begin{cases} -\Delta\psi_i = \gamma_i\psi_i & \text{in } \Omega \\ \partial_\nu\psi_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Theorem 3.8. *Let $\bar{n} \in \mathbb{N}$ be the largest index corresponding to an eigenvalue $\gamma_{\bar{n}} < (\lambda k - \mu\omega)/\mu$ and assume that $\bar{n} \geq 1$. Then, for any $1 \leq n \leq \bar{n}$, if γ_n is an eigenvalue of odd multiplicity, then there exists a maximal closed and connected subset $\mathscr{C}_n \subset \mathcal{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3) \times \mathbb{R}$ and $\beta_n \in (0, +\infty)$ such that*

(a) $\beta_n \frac{\lambda k - \mu\omega}{\mu\beta_n + 2k^2} = \gamma_n$ *and*

(b) *the corresponding constant non trivial solution* $(w_1, w_2, u) \in \mathscr{C}_n$

and either

- $\mathscr{C}_n$ *is unbounded in* β, *or*
- $\mathscr{C}_n$ *contains another point which satisfies (a) and (b) for a different value of* n.

Finally, on each continuum $\mathscr{C}_n$, β *is bounded from below away from zero.*

Remark 3.9. One could wonder what happens for the eigenvalue $\gamma_0 = 0$. The answer is actually already contained in the previous remarks: indeed, γ_0 corresponds to the value $\beta = 0$, and we have already observed in Lemma 3.3 that in this situation there exists a one dimensional subspace of constant solutions. Observe that, by the same argument of Theorem 3.8, this line should be unbounded in β, collapse in an other bifurcation point, or meet the boundary of the set of solutions (this case is excluded in Theorem 3.8 for the higher order bifurcation points). Since this branch of solutions is defined only for $\beta = 0$ only one between the last two possibility holds. In particular, this depends on our definition of solutions: indeed, either we allow solutions to be negative, or we restrict ourselves to the case of non-negative solutions. In this latter case, the more feasible for applications, the branch of solutions unraveling for β contains as endpoints the semi-trivial constant solutions.

Remark 3.10. Exploiting the symmetry of the domain Ω and of the eigenfunctions, we can also give a more detailed description of the branches in Theorem 3.8, in particular we can show that the symmetries of the eigenfunctions are preserved along a global branch of solutions, see for instance [Hea88].

Proof. The theorem follows from the bifurcation theorem by Rabinowitz, see [Rab71]. For $\beta > 0$ and a corresponding nontrivial constant solution (w_1, w_2, u) with $w_1 = w_2$, we look for a new solution of the form $(w_1 + \varphi_1, w_2 + \varphi_2, u + \varphi)$, for small perturbations $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi) \in X(\Omega)$. Inserting this ansatz in the system (3.1) we obtain

$$-\Delta\boldsymbol{\varphi} = A_\beta\boldsymbol{\varphi} + \begin{pmatrix} k\varphi_1\varphi - \beta\varphi_1\varphi_2 \\ k\varphi_2\varphi - \beta\varphi_1\varphi_2 \\ -\mu\varphi^2 - k(\varphi_1 + \varphi_2)\varphi \end{pmatrix} = A_\beta\boldsymbol{\varphi} + H(\beta, \boldsymbol{\varphi}) \qquad \text{in } \Omega$$

completed by homogeneous Neumann boundary conditions. Here the nonlinear functional $H : (\mathbb{R}, X) \to X$ is continuous and $\|H(\beta, \boldsymbol{\varphi})\|_X \leq C\|\boldsymbol{\varphi}\|_X^2$ for a constant $C > 0$ that can be chosen uniformly on compact sets of β. Let us now introduce the operator $L \in \mathcal{K}(X; X)$ defined as the linear map such that for any $u, f \in X$

$$u = Lf \Leftrightarrow \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \Omega. \end{cases}$$

We can rewrite the perturbed system as

$$\boldsymbol{\varphi} = (A_\beta + \mathrm{Id})L\boldsymbol{\varphi} + LH(\beta, \boldsymbol{\varphi}) = (A_\beta + \mathrm{Id})L\boldsymbol{\varphi} + h(\beta, \boldsymbol{\varphi}) \tag{3.3}$$

where now $h : (\mathbb{R}, X) \to X$ is a compact operator such that $\|h(\beta, \boldsymbol{\varphi})\|_X \leq C\|\boldsymbol{\varphi}\|_X^2$ and again the constant $C > 0$ that can be chosen uniformly on compact sets of β. We are now in a position to apply the bifurcation theorem by Rabinowitz [Rab71]: indeed, as a function of β, $(A_\beta + \mathrm{Id})L$ is a homotopy of compact operators, and a value $\bar{\beta}$ is a bifurcation point for the equation (3.3) whenever the set of solutions to the linear equation

$$\boldsymbol{\varphi} = (A_{\bar{\beta}} + \mathrm{Id})L\boldsymbol{\varphi}$$

has odd dimension. This translates to the system

$$\begin{cases} -\Delta\boldsymbol{\varphi} = A_{\bar{\beta}}\boldsymbol{\varphi} & \text{in } \Omega \\ \partial_\nu\boldsymbol{\varphi} = 0 & \text{on } \Omega. \end{cases}$$

The spectral properties of the matrix $A_{\bar{\beta}}$ where already studied in Lemma 3.4: the matrix has a unique positive eigenvalue $\bar{\beta}(\lambda k - \mu\omega)/(\mu\bar{\beta} + 2k^2)$ that correspond to the eigenvector $(-1, 1, 0)$. As a consequence, we have that (γ_i, ψ_i) is an eigenvalue-eigenvector couple of (3.2) and $\bar{\beta}(\lambda k - \mu\omega)/(\mu\bar{\beta} + 2k^2) = \gamma_i$ if and only if $\boldsymbol{\varphi} = (\psi_i, -\psi_i, 0)$ solves the previous system for the prescribed value of $\bar{\beta}$. In particular we obtain that if γ_i has odd multiplicity, then $\bar{\beta}$ is a bifurcation point in the sense of the thesis.

It remains to show that the continua $\mathscr{C}_n$ are either unbounded in β or meet the set $\mathscr{S}_\beta$ in another bifurcation point: recalling the bifurcation theorem by Rabinowitz [Rab71, Theorem 1.3], we already know that each continuum is either unbounded in $\mathbb{R} \times X(\Omega)$ or collapses on the set $\mathscr{S}_\beta$. As a result, it is sufficient for us to show that if the continuum $\mathscr{C}_n$ is unbounded, it must be unbounded in β component. First of all, let us observe that on each of the continuum $\mathscr{C}_n$, the parameter $\beta > 0$: indeed if there exists $(w_1, w_2, u) \in \mathscr{C}_n$ with $\beta = 0$, this solution must be one of the solutions found in Lemma 3.3. But then, by the stability analysis of the solutions and the observations in Remark 3.9, we know that the sets of solutions associated to $(0, 0, 0)$ and $(0, 0, \lambda/\mu)$ (which are defined for all β) are isolated, while the linear space of weakly stable solutions admits no subsequent bifurcations apart from its midpoint where the set $\mathscr{S}_\beta$ is generated, and we obtain a contradiction.

We now recall that, by Proposition 3.1, the non-negative solutions satisfy the system of inequalities

$$\begin{cases} w_1 \geq 0, w_2 \geq 0, 0 \leq u \leq \lambda/\mu \\ u + w_1 + w_2 \leq \dfrac{(\lambda + \omega)^2}{4\mu\omega} \end{cases}$$

and either all the inequalities are strict or the solution is constant. It follows that if β is bounded on $\mathscr{C}_n$, there must exists on $\mathscr{C}_n$ a solution which is constant. Discarding the solutions on $\mathscr{S}_\beta$, the only possibility, thanks to Lemma 3.4, are solutions which are either (case (a)) strongly unstable or (case (b) and (c)) strongly stable, and their stability does not depend on β: that is to say, no bifurcation point can belong to this set of solution. As before, this leads us to a contradiction. □

We can strengthen the conclusion of Theorem 3.8 by noticing that we are in a position to apply the analytic bifurcation theory developed by Dancer in [Dan71, Dan73] (see also [BT03, Theorem 9.1.1]). We have

Theorem 3.11. *Under the assumptions of Theorem 3.8, for any continua of solutions $\mathscr{C}_n$ there exists a curve $\mathfrak{C}_n := \{(B(s), \mathbf{W}(s)) : \mathbb{R} \mapsto \mathbb{R}^+ \times X\} \subset \mathscr{C}_n$, which contains the bifurcation point that spawns $\mathscr{C}_n$, such that*

- *at any point, the curve $\mathfrak{C}_n$ can be locally reparametrized as an analytic curve;*
- *the set of possible secondary bifurcation points on $\mathfrak{C}_n$ has no accumulation points.*

Moreover

- *either $\mathfrak{C}_n$ is a closed loop, and meets the set $\mathscr{S}_\beta$ in two distinct bifurcation points;*
- *or the set $\mathfrak{C}_n$ is unbounded in $\mathbb{R}^+ \times X$, and more specifically*

$$B(s) \to +\infty \qquad \text{as } s \to \infty$$

while, letting $(w_1(s), w_2(s), u(s)) = \mathbf{W}(s)$, the quantity

$$\|w_1(s)\|_{Lip(\overline{\Omega})} + \|w_2(s)\|_{Lip(\overline{\Omega})} + \|u(s)\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})}$$

is bounded uniformly in s for all $\alpha < 1$.

In the one-dimensional case we can say even more, specifically

Theorem 3.12. *Under the assumptions of Theorem 3.8, let us moreover suppose that $\Omega \subset \mathbb{R}$ is an open and bounded interval. Then any eigenvalue γ of (3.2) is of multiplicity one, and the corresponding continuum of solutions $\mathscr{C}_n$ (and $\mathfrak{C}_n$) generating from the set $\mathscr{S}_\beta$ at the value*

$$\beta_n \frac{\lambda k - \mu\omega}{\mu\beta_n + 2k^2} = \gamma_n$$

is unbounded and it intersects the set $\mathscr{S}_\beta$ only once.

Proof. The proof follows again the main ideas presented in [Rab71]. Under the assumptions, there exist $a < b \in \mathbb{R}$ such that $\Omega = (a, b) \subset \mathbb{R}$: we can explicitly compute the eigenvalues of (3.2), which are given by

$$\gamma_n := \left(\frac{\pi}{b-a} n\right)^2 \qquad \text{for any } n \in \mathbb{N}.$$

Based on the discussion in Theorem 3.8, any value of γ_n corresponds to a bifurcation point, even though the set $\mathscr{C}_0$ generating from $\gamma_0 = 0$ is given by a trivial linear subspace of constant solutions, which has already being completely characterized in Lemma 3.3.

Let us consider, for a fixed γ_n with $n \geq 1$ as before, the continuum of solutions $\mathscr{C}_n$ that generates from the set $\mathscr{S}_\beta$. By the perturbations analysis conducted in Theorem 3.8, we know that the solutions are of the form

$$(w_{1,\beta}, w_{2,\beta}, u_\beta) = \left(\frac{\lambda k - \mu\omega}{\mu\beta_n + 2k^2}, \frac{\lambda k - \mu\omega}{\mu\beta_n + 2k^2}, \frac{\lambda\beta_n + 2k\omega}{\mu\beta_n + 2k^2}\right) + \varepsilon(\psi_n, -\psi_n, 0) + o(\varepsilon)$$

where ε is a parameter such that $\varepsilon \to 0$ when $\beta \to \beta_n$, ψ_n is a normalized eigenfunction of 3.2 in $\Omega = (a, b)$ and $o(\varepsilon)$ is a perturbation in $\mathcal{C}^{2,\alpha}([a, b])$ of order less than ε. In particular, letting

$$v_{\beta,n} = w_{1,\beta} - w_{2,\beta} = 2\varepsilon\psi_n + o(\varepsilon)$$

(where we have highlighted the index n of the eigenfunction which spawns $v_{\beta,n}$) we have that $v_{\beta,n}$ solves

$$\begin{cases} -v''_{\beta,n} = (-\omega + ku_\beta)v_{\beta,n} & \text{in } (a, b), \\ v'_{\beta,n}(a) = v'_{\beta,n}(b) = 0. \end{cases}$$

As a result, when ε is small, v_β has exactly n distinct simple zeroes in (a, b), located closely to the zeroes of the eigenfunction ψ_n. We recall that the solutions of the system (3.1) are bounded in $Lip([a, b])$ uniformly with respect to β and in particular the last component u_β is bounded in $\mathcal{C}^{2,\alpha}([a, b])$ for all $\alpha < 1$: it follows that there exists a parametrization of the continuum $\mathscr{C}_n$ with respect to which the functions $v_{\beta,n}$ vary smoothly and they also are uniformly bounded in $\mathcal{C}^{2,\alpha}([a, b])$ for all $\alpha < 1$.

We claim that on each continuum of solutions $\mathscr{C}_n$, the number of zeroes of the function $v_{\beta,n}$ does not change. To prove the claim, we first observe that, since the solutions depend smoothly in $\mathcal{C}^{2,\alpha}([a, b])$ on the parametrization, if $v_{\beta,n}$ changes the number of zeros, there exists a solution v inside of $\mathscr{C}_n$ that has a zeroes of multiplicity at least two, which could be at an interior point or at the boundary (this is thanks to the homogeneous Neumann condition). But then the uniqueness theorem for ordinary differential equations with smooth coefficients would imply that the function v must be equal to 0. As a result, the corresponding solution (w_1, w_2, u) has equal first and second component: reasoning as in Lemma 3.3, by letting $V = w_1 + w_2$, we obtain a solution to

$$\begin{cases} -\Delta V = -\omega V + kVu - \beta V^2 & \text{in } \Omega \\ -\Delta u = \lambda u - \mu u^2 - kVu & \text{in } \Omega \\ \partial_\nu V = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

and again thanks to the results in [Mim79], the solution (w_1, w_2, u) must be a constant solution. We have two possibilities:

1) The parameter β corresponding to (w_1, w_2, u) is 0. In this case, combining Lemma 3.3 and Remark 3.9, we already know that from $\beta = 0$ a linear space of solutions is generated, and on this continuum there are no subsequent bifurcations, and thus we are lead to a contradiction;

2) The parameter $\beta > 0$: in this case (w_1, w_2, u) belongs necessarily to the set $\mathscr{S}_\beta$ (we recall that if $\beta > 0$ the only bifurcation points belong to $\mathscr{S}_\beta$, see Remark 3.5). From the previous discussion it must be that the point corresponds to a different eigenvalue γ_m, $m \neq n$, and locally the solutions can be written as a perturbation along the line direction spanned by the eigenfunction ψ_m. Similarly as before, the difference of the first two components can be asymptotically expanded as

$$v_{\beta,m} = w_{1,\beta} - w_{2,\beta} = 2\varepsilon\psi_m + o(\varepsilon)$$

and, again, for $\varepsilon \to 0$, the solution $v_{\beta,m}$ has m distinct simple zeroes on (a, b). In particular, the solution must have $m \neq n$ zeroes in a neighborhood of the bifurcation point, leading us to a contradiction. □

We want to study more closely the bifurcation branches $\mathscr{C}_n$ that are unbounded in $\mathbb{R} \times X$. We recall that assumptions (H) holds, in particular $\lambda k > \mu\omega$. Let us consider the set, denote by $\mathscr{P}$, of all solutions $\mathbf{w}_\beta = (w_{1,\beta}, w_{2,\beta}, u_\beta)$ of (3.1) with competition parameter $\beta > 0$ such that all of its components are strictly positive. We use a blow-up technique first introduced in [DD94] to study a similar situation.

Lemma 3.13. *There exists $M > 0$ such that*

$$\frac{1}{M}\|w_{2,\beta}\|_{L^\infty(\Omega)} \leq \|w_{1,\beta}\|_{L^\infty(\Omega)} \leq M\|w_{2,\beta}\|_{L^\infty(\Omega)}$$

for all $\mathbf{w}_\beta \in \mathscr{P}$ and β sufficiently large.

Proof. We argue by contradiction, assuming that there exists a sequence of solutions in $\mathscr{P}$ that invalidates the thesis. Without loss of generality, let us assume that $\|w_{1,n}\|_{L^\infty(\Omega)} \leq \|w_{2,n}\|_{L^\infty(\Omega)}$ and that the ratio $\|w_{1,n}\|_{L^\infty(\Omega)}/\|w_{2,n}\|_{L^\infty(\Omega)} \to 0$ as $\beta_n \to +\infty$. Let us introduce the renormalized functions

$$\bar{w}_{i,n} = \frac{w_{i,n}}{\|w_{i,n}\|_{L^\infty(\Omega)}} \qquad \text{for } i = 1, 2$$

which are solutions to

$$\begin{cases} -\Delta\bar{w}_{1,n} = -\omega\bar{w}_{1,n} + k\bar{w}_{1,n}u - \beta_n\|w_{2,n}\|_{L^\infty(\Omega)}\bar{w}_{1,n}\bar{w}_{2,n} & \text{in } \Omega \\ -\Delta\bar{w}_{2,n} = -\omega\bar{w}_{2,n} + k\bar{w}_{2,n}u - \beta_n\|w_{1,n}\|_{L^\infty(\Omega)}\bar{w}_{1,n}\bar{w}_{2,n} & \text{in } \Omega \\ -\Delta u_n = \lambda u_n - \mu u_n^2 - k(w_{1,n} + w_{2,n})u_n & \text{in } \Omega \\ \partial_\nu\bar{w}_{i,n} = \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

We distinguish between two different cases.

1) $\beta_n\|w_{2,n}\|_{L^\infty(\Omega)} \leq C$. In this case, all the terms in the equations are bounded uniformly in β_n, and thus in it easy to see that the sequence $\bar{w}_{i,n}$, u_n and also $w_{i,n}$, are uniformly bounded in $W^{2,p}(\Omega)$ for any $p < \infty$. Up to a subsequence, we deduce strong convergence of the renormalized densities to some limit profile $(\bar{w}_{1,\infty}, \bar{w}_{2,\infty}, u_\infty)$ for both $\bar{w}_{1,\infty}$ and $\bar{w}_{2,\infty}$ non trivial, while by assumption $w_{i,n} \to 0$ uniformly in Ω. Moreover, by assumption we have that

$$\beta_n\|w_{2,n}\|_{L^\infty(\Omega)} \to C \geq 0 \quad \text{while } \beta_n\|w_{1,n}\|_{L^\infty(\Omega)} \to 0.$$

As a result, the limit profiles

$$\begin{cases} -\Delta\bar{w}_{1,\infty} = -\omega\bar{w}_{1,\infty} + k\bar{w}_{1,\infty}u - C\bar{w}_{1,\infty}\bar{w}_{2,\infty} & \text{in } \Omega \\ -\Delta\bar{w}_{2,\infty} = -\omega\bar{w}_{2,\infty} + k\bar{w}_{2,\infty}u & \text{in } \Omega \\ -\Delta u_\infty = \lambda u_\infty - \mu u_\infty^2 & \text{in } \Omega \\ \partial_\nu\bar{w}_{i,\infty} = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, we have that the equation in u_∞ has only the constant solutions $u_\infty = 0$ or λ/μ. Inserting this information in the equation satisfied by $\bar{w}_{2,\infty}$ we see that

$$\begin{cases} -\Delta\bar{w}_{2,\infty} = -\omega\bar{w}_{2,\infty} + k\bar{w}_{2,\infty}u_\infty = C'w_{2,\infty} & \text{in } \Omega \\ \partial_\nu\bar{w}_{2,\infty} = 0 & \text{on } \partial\Omega. \end{cases}$$

where the constant C' is non zero by the assumption (H). It follows that necessarily $\bar{w}_{2,\infty} \equiv 0$, in contradiction with $\|\bar{w}_{2,\infty}\|_{L^\infty(\Omega)} = 1$.

2) $\beta_n\|w_{2,n}\|_{L^\infty(\Omega)} \to +\infty$. Let us test the equation in $\bar{w}_{i,n}$ by $\bar{w}_{i,n}$ itself. Recalling that $\bar{w}_{i,n} \geq 0$ and that $u_n \leq \lambda/\mu$ we have

$$\int_\Omega |\nabla\bar{w}_{i,n}|^2 \leq k\frac{\lambda}{\mu}|\Omega|,$$

where $|\Omega|$ is the measure of the set Ω. As a result, $\bar{w}_{i,n}$ are bounded uniformly in $H^1(\Omega)$ and thus converge to some weak limit $\bar{w}_{i,\infty} \in H^1(\Omega)$. Moreover, the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ yields $\bar{w}_{i,n} \to \bar{w}_{i,\infty}$ strongly in $L^2(\Omega)$ and further, since by construction $\|w_{i,n}\|_{L^\infty(\Omega)} = 1$, we have that $\bar{w}_{i,n} \to \bar{w}_{i,\infty}$ strongly in $L^p(\Omega)$ for any $p \geq 2$. Recalling that the equation in u_n contains only uniformly bounded terms, up to a subsequence we have $u_n \to u_\infty$ in $W^{2,p}(\Omega)$ for any $p < \infty$. Let us show that each component of the limit configuration $(\bar{w}_{1,\infty}, \bar{w}_{2,\infty}, u_\infty)$ is non trivial: from the equations satisfied by $\bar{w}_{i,n}$ we have that

$$\begin{cases} -\Delta \bar{w}_{i,n} + \omega \bar{w}_{i,n} \leq k \bar{w}_{i,n} u_n & \text{in } \Omega \\ \partial_\nu \bar{w}_{i,n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, letting $g_{i,n} \in H^1(\Omega)$ be the solution to

$$\begin{cases} -\Delta g_{i,n} + \omega g_{i,n} = k \bar{w}_{i,n} u_n & \text{in } \Omega \\ \partial_\nu g_{i,n} = 0 & \text{on } \partial\Omega \end{cases}$$

we have, from the previous discussion, that the sequence $\{g_{i,n}\}_n$ is compact in $W^{2,p}(\Omega)$ for any $p > 1$ and, in particular, in $\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$. The maximum principle, on the other hand, yields $0 \leq \bar{w}_{i,n} \leq g_{i,n}$. Let us then assume, by contradiction, that either $\bar{w}_{i,\infty} = 0$ or $u_\infty = 0$: then it follows $g_{i,n} \to 0$ uniformly, that is, $\bar{w}_{i,n} \to 0$ uniformly, in contradiction with $\|w_{i,n}\|_{L^\infty(\Omega)} = 1$.

Testing the equations in $\bar{w}_{1,n}$ by $\varphi \in H^1(\Omega)$, we have

$$\beta_n \|w_{2,n}\|_{L^\infty(\Omega)} \int_\Omega \bar{w}_{1,n} \bar{w}_{2,n} \varphi = \int_\Omega (k u_n - \omega) \bar{w}_{1,n} \varphi - \int_\Omega \nabla w_{1,n} \cdot \nabla \varphi \leq C \tag{3.4}$$

so that, using our assumption

$$\beta_n \|w_{1,n}\|_{L^\infty(\Omega)} \int_\Omega \bar{w}_{1,n} \bar{w}_{2,n} \varphi = \frac{\|w_{1,n}\|_{L^\infty(\Omega)}}{\|w_{2,n}\|_{L^\infty(\Omega)}} \cdot \beta_n \|w_{2,n}\|_{L^\infty(\Omega)} \int_\Omega \bar{w}_{1,n} \bar{w}_{2,n} \varphi \to 0.$$

As a result, $\bar{w}_{2,\infty}$ is a weak solution of the equation

$$\begin{cases} -\Delta \bar{w}_{2,\infty} = -\omega \bar{w}_{2,\infty} + k \bar{w}_{2,\infty} u_\infty & \text{in } \Omega \\ \partial_\nu \bar{w}_{2,\infty} = 0 & \text{on } \partial\Omega \end{cases}$$

where $0 \leq u_\infty \leq \lambda/\mu$. By the maximum it follows that either $\bar{w}_{2,\infty} \equiv 0$ or $\bar{w}_{2,\infty} \geq C > 0$. The first case was already excluded, thus the latter holds. But then equation (3.4), with $\varphi = 1$, yields

$$\beta_n \|w_{2,n}\|_{L^\infty(\Omega)} \cdot \int_\Omega \bar{w}_{1,n} \bar{w}_{2,n} \leq C \implies \int_\Omega \bar{w}_{1,\infty} \bar{w}_{2,\infty} = 0$$

which implies $\bar{w}_{1,\infty} = 0$, in contradiction with the previous discussion. □

Lemma 3.14. *The set $\mathscr{P}$ is a pre-compact subset of $\mathcal{C}^{0,\alpha} \times \mathcal{C}^{0,\alpha} \times \mathcal{C}^{2,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. Moreover any converging subsequence $(w_{1,n}, w_{2,n}, u_n) \to (w_{1,\infty}, w_{2,\infty}, u_\infty)$ with $\beta_n \to +\infty$ is such that*

- *either $(w_{1,\infty}, w_{2,\infty}, u_\infty)$ has all non trivial components and, letting $V = w_{1,\infty} - w_{2,\infty}$, V changes sign and $(V, u_\infty) \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ is a non-trivial solution of*

$$\begin{cases} -\Delta V = -\omega V + kVu & \text{in } \Omega \\ -\Delta u = \lambda u - \mu u^2 - k|V|u & \text{in } \Omega \\ \partial_\nu V = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

- *or*

$$(\beta_n w_{1,n}, \beta_n w_{2,n}, u_n) \sim \left(\frac{\lambda k - \mu\omega}{\mu}, \frac{\lambda k - \mu\omega}{\mu}, \frac{\lambda}{\mu} \right)$$

as $n \to +\infty$in $Lp(\Omega)$ for any $p < \infty$ and weakly in $H^1(\Omega)$.

Proof. The compactness in strong topology of the sequence of solutions was already established in Proposition 3.1, we are left with the study of the asymptotic profiles. First of all we exclude that case $u_n \to 0$ (which would hold uniformly in Ω as by the compactness properties): in this situation, indeed, we would have

$$\begin{cases} -\Delta w_{i,n} = -\omega w_{i,n} + k w_{i,n} u_n - \beta_n w_{i,n} w_{j,n} \leq -\frac{\omega}{2} w_{i,n} & \text{in } \Omega \\ \partial_\nu w_{i,n} = 0 & \text{on } \partial\Omega \end{cases}$$

for n sufficiently large, which implies that necessarily $w_{i,n} \equiv 0$ for n large, against the assumptions.

Let us now assume that

$$w_{1,n}, w_{2,n} \to 0 \qquad \text{uniformly in } \Omega.$$

Passing to the limit in n the equation in u_n, we see that u_∞ satisfies

$$\begin{cases} -\Delta u_\infty = \lambda u_\infty - \mu u_\infty^2 & \text{in } \Omega \\ \partial_\nu u_\infty = 0 & \text{on } \partial\Omega \end{cases}$$

which implies that $u_n \to \lambda/\mu$ in $\mathcal{C}^{2,\alpha}(\Omega)$ (recall that we have already excluded the case $u_n \to 0$). We introduce the renormalized functions

$$\bar{w}_{i,n} := \frac{w_{i,n}}{\|w_{1,n}\|_{L^\infty(\Omega)}}$$

which are solutions to

$$\begin{cases} -\Delta \bar{w}_{1,n} = -\omega \bar{w}_{1,n} + k\bar{w}_{1,n} u_n - \beta_n \|w_{1,n}\|_{L^\infty(\Omega)} \bar{w}_{1,n}\bar{w}_{2,n} & \text{in } \Omega \\ -\Delta \bar{w}_{2,n} = -\omega \bar{w}_{2,n} + k\bar{w}_{2,n} u_n - \beta_n \|w_{1,n}\|_{L^\infty(\Omega)} \bar{w}_{1,n}\bar{w}_{2,n} & \text{in } \Omega \\ \partial_\nu \bar{w}_{i,n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us observe that, thanks to Lemma 3.13, we have that $\bar{w}_{i,n}$ are bounded by some positive constant $M > 0$; moreover, using the same initial steps of Case 2) in Lemma 3.13, we see that $\bar{w}_{i,n} \to \bar{w}_{i,\infty}$ in $L^p(\Omega)$ for any $p < \infty$ and weakly in $H^1(\Omega)$ and also $\bar{w}_{i,\infty} \neq 0$. Letting $V_n = \bar{w}_{1,n} - \bar{w}_{2,n}$, we have that $\|V_n\|_{L^\infty(\Omega)} \leq M + 1$ and

$$\begin{cases} -\Delta V_n = (-\omega + k u_n) V_n & \text{in } \Omega \\ \partial_\nu V_n = 0 & \text{on } \partial\Omega. \end{cases}$$

As a result of the strong convergence $u_n \to \lambda/\mu$, we have that $V_n \to V_\infty$ in $\mathcal{C}^{2,\alpha}(\Omega)$, solution of the limit equation

$$\begin{cases} -\Delta V_\infty = \frac{\lambda k - \mu\omega}{\mu} V_\infty & \text{in } \Omega \\ \partial_\nu V_\infty = 0 & \text{on } \partial\Omega \end{cases}$$

where, by assumption (H), $(\lambda k - \mu\omega)/\mu > 0$. Consequently either $V_\infty \equiv 0$ or (if $(\lambda k - \mu\omega)/\mu$ is a positive eigenvalue of the Laplacian with Neumann boundary conditions) V_∞ changes necessarily sing, and thus $\bar{w}_{1,\infty} \neq 0$ and $\bar{w}_{2,\infty} \neq 0$. Testing the equation in $\bar{w}_{i,n}$ by $\bar{w}_{i,n}$ itself, we find

$$\int_\Omega |\nabla \bar{w}_{i,n}| + \beta_n \|w_{1,n}\|_{L^\infty(\Omega)} \int_\Omega \bar{w}_{i,n}^2 \bar{w}_{j,n} = \int_\Omega (-\omega + k u_n)\, \bar{w}_{i,n}^2 \leq C$$

which implies, in particular, that $\beta_n \|w_{1,n}\|_{L^\infty(\Omega)} \to C$ for some constant $C \geq 0$. Finally, passing to weak limit the equation in $\bar{w}_{i,n}$ we find

$$\begin{cases} -\Delta \bar{w}_\infty = \left(-\omega + k\frac{\lambda}{\mu}\right) \bar{w}_\infty - C\bar{w}_\infty^2 & \text{in } \Omega \\ \partial_\nu \bar{w}_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

If $C = 0$, since $(\lambda k - \mu\omega)/\mu > 0$ and $\bar{w}_\infty$ is by the maximum principle non negative, it must be $\bar{w}_\infty \equiv 0$, in contradiction with the renormalization. As a result $C > 0$ and, from a direct application of the maximum principle (see also Lemma 3.15 below), we have that the only non negative solution to the previous equation are the constant. In particular, it must be $\bar{w}_\infty \equiv 1$, thus

$$\left(-\omega + k\frac{\lambda}{\mu}\right) - C = 0 \implies C = \frac{\lambda k - \mu\omega}{\mu}. \qquad \square$$

Lemma 3.15. *Let $\Omega \subset \mathbb{R}^n$ a smooth domain, a and b positive constants. If $u \in H^1(\Omega)$ is a non negative solution to*

$$\begin{cases} -\Delta u = (a - bu)u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

then $u \equiv 0$ or $u \equiv a/b$.

Proof. Letting $w = u - a/b$, we have

$$\begin{cases} -\Delta w = -buw & \text{in } \Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases}$$

Testing the equation by w itself and recalling the assumption $u \geq 0$ and $b > 0$ we have

$$\int_\Omega |\nabla w|^2 + b \int_\Omega u w^2 = 0$$

and we easily conclude that w has to be constant (hence u) and either $u = 0$ or $u = a/b$. $\square$

As usual, in the one-dimensional case we can say more, and in particular we can show the following stronger version of Lemma 3.14.

Lemma 3.16. *Let $\Omega = (a, b)$ with $a < b \in \mathbb{R}$. The set $\mathscr{P}$ is a pre-compact subset of $\mathcal{C}^{0,\alpha} \times \mathcal{C}^{0,\alpha} \times \mathcal{C}^{2,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. Moreover any converging subsequence $(w_{1,n}, w_{2,n}, u_n) \to (w_{1,\infty}, w_{2,\infty}, u_\infty)$ with $\beta_n \to +\infty$ is such that*

- *either $(w_{1,\infty}, w_{2,\infty}, u_\infty)$ has all non trivial components and, letting $V = w_{1,\infty} - w_{2,\infty}$, V changes sign and $(V, u_\infty) \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ is a non-trivial solution of*

$$\begin{cases} -V'' = -\omega V + kVu & \text{in } \Omega \\ -u'' = \lambda u - \mu u^2 - k|V|u & \text{in } \Omega \\ V'(a) = V'(b) = u'(a) = u'(b) = 0 \end{cases}$$

- *or*

$$(w_{1,n}, w_{2,n}, u_n) = \left(\frac{\lambda k - \mu\omega}{\beta_n \mu}, \frac{\lambda k - \mu\omega}{\beta_n \mu}, \frac{\lambda}{\mu} \right)$$

as $n \to +\infty$.

Proof. As in Theorem 3.12, we can use the auxiliary function $v_\beta = w_{1,\beta} - w_{2,\beta}$ to study more accurately the second case of the lemma. The conclusion is reached once again by counting the number of zeros of v_β and observing that this must be constant on each bifurcation branch. $\square$

As a direct consequence, we have that there exists $\delta > 0$ such that, for $\bar{\beta}$ sufficiently large,

$$\operatorname{dist}(\mathscr{P} \setminus \mathscr{S}_\beta, \mathscr{S}_\beta) > \delta \qquad \text{for all } \beta \geq \bar{\beta}$$

where the distance is taken in the sense of the $\mathcal{C}^{0,\alpha} \times \mathcal{C}^{0,\alpha} \times \mathcal{C}^{2,\alpha}(\overline{\Omega})$ norm for any $\alpha \in (0,1)$. Moreover each branch of solutions constructed in Theorem 3.12 converge (up to a subsequence) to a disjoint set of solutions for the limit problem, characterized by the different number of zeroes for the function V.

4. Optimal repartition of resources

We continue the investigation of the model by addressing an interesting application: can the model be used to predict the optimal repartition of the domain Ω in hunting territories, that is, the optimal number of packs?

To answer this question, we first focus on the limit stationary system satisfied by the densities in the case of segregation. We shall prove two complementary results in this scenario:

- firstly, we show that each bounded domain $\Omega \Subset \mathbb{R}^N$ can sustain a maximum number of densities of predators (see Lemma 4.2 and Theorem 4.5). This immediately implies that there exists a number $k \geq 1$ of packs that, for a given configuration of parameters, maximizes the total population of predators;
- secondly, we show that, under particular choices of the parameters, the total population of predators in the case of two packs is strictly higher than the of only one pack, implying that in these cases the optimal configuration is given by a finite but strictly greater than two number of packs.

We start by proving a bound on the total number of packs that can be sustain by a domain Ω. Let us recall, in this regard, Proposition 3.1 (which can be generalized to the case of N densities of predators).

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain, and let β, D, d_i, ω_i, k_i, $a_{ij} = a_{ji}$ for $1 \le i, j \le k$ be positive parameters, we consider the solutions $\mathbf{w} = (w_1, \dots, w_N, u) \in \mathcal{C}^{2,\alpha}(\Omega)$ of the system*

$$
\begin{cases}
-d_i \Delta w_i = \left(-\omega_i + k_i u - \mu_i w_i - \beta \sum_{j \neq i} a_{ij} w_j\right) w_i \\
-D \Delta u = \left(\lambda - \mu u - \sum_{i=1}^N k_i w_i\right) u \\
\partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.1}
$$

Then $\mathbf{w}$ are uniformly bounded in $L^\infty(\Omega)$ with respect to $\beta > 0$ and moreover there exists C (independent of β) such that

$$\|(w_1, \dots, w_N)\|_{Lip(\overline{\Omega})} + \|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \le C$$

If $\{\mathbf{w}_\beta\}_\beta$ is a family of solution as above, defined for $\beta \to +\infty$, then, up to subsequences, there exists $\mathbf{w} = (w_1, \dots, w_N, u)$ with $(w_1, \dots, w_N) \in Lip(\bar{\Omega})$ and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and

$$(w_{1,\beta}, \dots, w_{N,\beta}) \to (w_1, \dots, w_N) \text{ in } \mathcal{C}^{0,\alpha} \cap H^1(\overline{\Omega}), u_\beta \to u \text{ in } \mathcal{C}^{2,\alpha}(\overline{\Omega}).$$

Any limit satisfies the system of inequalities (in the sense of measures)

$$
\begin{cases}
-d_i \Delta w_i \le (-\omega_i + k_i u) w_i \\
-\Delta \left(d_i w_i - \displaystyle\sum_{j \neq i} d_j w_j \right) \ge (-\omega_i + k_i u) w_i - \displaystyle\sum_{j \neq i} (-\omega_j + k_j u) w_j & \text{in } \Omega \\
-D \Delta u = \left(\lambda - \mu u - \displaystyle\sum_{i=1}^k k_i w_i \right) u \\
\partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.2}
$$

Finally, the subset $\{x \in \Omega : \sum_{i=1}^N w_i = 0\}$ is a rectifiable set of codimension 1, made of the union of a finite number of $\mathcal{C}^{1,\alpha}$ smooth sub-manifolds.

The proofs of this result follows the same general ideas of Proposition 3.1, and it is then omitted. We point out that the conclusion on the regularity of the limit free-boundary problem follows directly from the main results in [CTV05b], [CKL09] and [TT12]. In order to simplify the exposition, we shall assume $a_{ij} = 1$ for all $1 \le i, j \le k$: the results that follow can be generalized without any real effort. A much harder case is when the competition matrix is not symmetric, that is when $a_{ij} \neq a_{ji}$ for some $i \neq j$: even though most of the results are valid also in this case, we will not consider it here, since we can only obtain a less complete description of the solutions, but we refer the reader to [STVZ] to understand the new difficulties. Before we continue, we need some uniform assumptions on the coefficients in (4.2): in particular all the coefficients $D, d_1, \dots, d_N, \omega_1, \dots, \omega_N, k_1, \dots, k_N$ are positive and uniformly bounded from zero and infinity, that is, there exists $\delta > 0$ such that

$$\delta < D, d_1, \dots, d_N, \omega_1, \dots, \omega_N, k_1, \dots, k_N < \frac{1}{\delta}.$$

Evidently the previous assumptions are trivially verified for a fixed number N of densities, but are needed when studying the model for an a priori unspecified number of densities.

We start with the following result, which states that for each environment Ω there is a maximal number of densities of predators $\bar{N}$ that can be sustained.

Lemma 4.2. *For a given smooth domain $\Omega \subset \mathbb{R}^N$, there exists $\bar{N} \in \mathbb{N}$ such that any non negative solution $(w_1, \dots, w_N, u) \in H^1(\Omega)$ of (4.2) has at most $\bar{N} + 1$ non trivial components.*

In order to prove the previous result, we need to recall the notion of optimal partition (see for instance [HHOT09] for a general survey and some fundamental results). Caveat: for consistency with the theory of optimal partitions, in the next two results eigenvalues will be counted starting from the index 1. For any $1 \le N \in \mathbb{N}$ we say that a family $\mathcal{D} = \{D_1, \dots, D_N\}$ of subsets of Ω is a N-partition of Ω if

$$D_i \cap D_j = \emptyset \ \forall i \neq j \text{ and } \cup_{i=1}^N D_N = \Omega.$$

For each D_i, we define the generalized first eigenvalue as

$$\gamma_1(D_i) := \inf\left\{ \int_\Omega |\nabla u|^2 \Big/ \int_\Omega u^2 : u \in H^1(\Omega),\ u = 0 \text{ in } \Omega \setminus D_i \right\}$$

and for the partition $\mathcal{D}$ we assign the total value

$$\Lambda(\mathcal{D}) = \max_i \gamma_1(D_i).$$

A partition $\mathcal{D}$ is optimal if it minimize the value of $\Lambda(\mathcal{D})$ among all N-partitions. We recall the following result (see [HHOT09, Corollary 5.6]), which follows from the Courant-Fischer characterization of the eigenvalues of compact hermitian operators.

Theorem 4.3. *Let $\gamma_N(\Omega)$ be the N-th eigenvalue (counted with multiplicity) of*

$$\begin{cases} -\Delta u = \gamma u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\Lambda(\mathcal{D}) \geq \gamma_N(\Omega)$$

for all N-partitions $\mathcal{D}$ of Ω.

Proof of Lemma 4.2. If the component u is zero, all the components of the solution are zero. Indeed, testing the inequalities in (4.2) by $w_i \in Lip(\overline{\Omega})$ we obtain

$$\int_\Omega d_i |\nabla w_i|^2 + (\omega_i - k_i u) w_i^2 = 0$$

which yields the claim taking into account that $u \equiv 0$. As a result, we can assume $u \geq 0$ and $u \neq 0$, that is, by the maximum principle, $u > 0$ in $\overline{\Omega}$. Let us now turn to the equation in u: since $w_i \geq 0$ for $i = 1, \dots, k$, we have

$$-D\Delta u = \left(\lambda - \mu u - \sum_{i=1}^N k_i w_i\right) u \leq (\lambda - \mu u)\, u \implies u \leq \frac{\lambda}{\mu}.$$

On the other hand, we have

$$-d_i \Delta w_i = (-\omega_i + k_i u)\, w_i \leq \left(-\omega_i + k_i \frac{\lambda}{\mu}\right) w_i$$

that is, letting $\Omega_i := \{w_i > 0\}$, w_i is a sub-solution to

$$\begin{cases} -d_i \Delta w_i \leq \left(-\omega_i + k_i \frac{\lambda}{\mu}\right) w_i & \text{in } \Omega_i \\ w_i = 0 & \text{on } \partial\Omega_i \cap \Omega \\ \partial_\nu w_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \end{cases}$$

Considering the first eigenvalue of Ω_i, that is the minimal solution $\gamma_1(\Omega)$ of

$$\begin{cases} -\Delta \varphi_i = \gamma_1(\Omega_i) \varphi_i & \text{in } \Omega_i \\ \varphi_i = 0 & \text{on } \partial\Omega_i \cap \Omega \\ \partial_\nu \varphi_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \end{cases}$$

with $\varphi_i \neq 0$, by standard arguments, we have that $\gamma_1(\Omega_i) \geq 0$ and $\varphi_i > 0$ in Ω_i; by the comparison principle, it follows that

$$\frac{\lambda k_i - \mu \omega_i}{d_i \mu} < \gamma_1(\Omega_i) \implies w_i \equiv 0.$$

In particular we have that if all the components $w_1, \dots, w_N \neq 0$ then necessarily

$$\max_{i=1,\dots,k} \gamma_1(\Omega_i) < \max_{i=1,\dots,k} \frac{\lambda k_i - \mu \omega_i}{d_i \mu} = \bar{\gamma}.$$

Since $\Omega_1, \dots, \Omega_N$ is a N-partition of the set Ω, we evince by Theorem 4.3 that necessarily

$$\gamma_N(\Omega) < \bar{\gamma}.$$

As a result, we reach the desired conclusion recalling that the sequence of eigenvalues $\gamma_1 < \gamma_2 \leq \gamma_3 \leq \dots$ is unbounded. □

Remark 4.4. Using Weyl's asymptotic law for the Neumann Laplacian, we can obtain a more explicit bound on the constant $\bar{N}$. Indeed, for a fixed domain $\Omega \subset \mathbb{R}^2$ (similar estimates hold in any dimension), if we let $N(\gamma)$ stand for the number of eigenvalues for the Laplace operator with homogeneous Neumann boundary conditions in Ω which are less than γ, it can be shown that

$$N(\gamma) = \frac{|\Omega|}{4\pi}\gamma + \frac{|\partial\Omega|}{4\pi}\sqrt{\gamma} + o(\sqrt{\gamma}).$$

As a result, for a fixed domain Ω we can obtain the following explicit estimate

$$\bar{N} \lesssim \frac{|\Omega|}{4\pi} \max_{i=1,\dots,k} \frac{\lambda k_i - \mu\omega_i}{d_i\mu} \qquad \text{for large values of } \max_{i=1,\dots,k} \frac{\lambda k_i - \mu\omega_i}{d_i\mu}.$$

This estimate confirms the intuition that by doubling the size of the domain Ω, we can in principle allow for twice the number of groups of predators.

We can now extend the result in Lemma 4.2 to the original competitive system. For any $N \in \mathbb{N}$ and $\beta > 0$, let us consider the set $\mathscr{P}_{N,\beta}$ of all solutions $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta)$ of (3.1) with competition parameter β such that all of its components are strictly positive. We recall that, by Proposition 4.1, the set of solutions of (3.1) is pre-compact in $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$.

Theorem 4.5. *For a given smooth domain $\Omega \subset \mathbb{R}^N$, there exist $\bar{N} \in \mathbb{N}$ and $\bar{\beta} > 0$ such if $k > \bar{N}$ and $\beta > \bar{\beta}$ then $\mathscr{P}_{N,\beta}$ contains only solutions of the form*

$$\|u_\beta - \lambda/\mu\|_{C^{2,\alpha}(\Omega)} + \|(w_{1,\beta}, \dots, w_{N,\beta})\|_{\mathcal{C}^{0,\alpha}(\Omega)} = o_\beta(1)$$

for every $\alpha \in (0, 1)$.

Proof. The statement will follow from some approximation results in combination with Lemma 4.2. First of all, we want to show that for $k > \bar{N}$ and β sufficiently large, then the solutions have to converge to the trivial solutions $(0, \dots, 0, 0)$ or $(0, \dots, 0, \lambda/\mu)$. We have

Claim. *Let $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta) \in H^1(\Omega)$ be a family of solutions to* (4.1) *and let us assume that there exists a solution $(w_1, \dots, w_N, u) \in H^1(\Omega)$ to* (4.2) *with $h+1$ non trivial components (with $1 \leq h \leq k$) such that $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta) \to (w_1, \dots, w_N, u)$ in $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$. Then there exists $\bar{\beta} > 0$ sufficiently large such that $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta)$ has exactly $h+1$ non-trivial components for $\beta \geq \bar{\beta}$.*

Let us first show how to use the claim in order to prove the first part of the theorem. Let us assume that we are given a family of solutions $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta) \in H^1(\Omega)$ to the system (4.1), with $\beta \to +\infty$. By Lemma 4.2 we already know that any solution of (4.2) has at most $\bar{N}+1$ non trivial components: let us assume that $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta)$ contains a sub-family (which we shall not relabel) with $\beta \to +\infty$ that has more than $\bar{N}+1$ non trivial components. Proposition 4.1 implies that, up to a subsequence, $(w_{1,\beta}, \dots, w_{N,\beta}, u_\beta) \to (w_1, \dots, w_N, u)$ in $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$, where $(w_1, \dots, w_N, u)$ solves the limit system (4.2), and thus has at most $\bar{N}+1$ components, in contradiction with our claim.

We now show the claim, arguing by contradiction and adopting the scheme of Lemma 3.13. Let $\mathbf{w}_n := (w_{1,\beta_n}, \dots, w_{N,\beta_n}, u_{\beta_n})$ be any sequence satisfying the assumptions of the claim and let $\mathbf{w} = (w_1, \dots, w_N, u)$ be its limit for $\beta_n \to +\infty$: by the maximum principle, if $\mathbf{w}$ has $h+1$ non trivial components, then necessarily u is strictly positive. Up to a relabelling, we can assume that the first h components $(w_1, \dots, w_h)$ are also non trivial, while $(w_{h+1}, \dots, w_N)$ are equal to zero. As a result, the sub-vector $(w_1, \dots, w_h, u)$ satisfies the conclusions of Proposition 4.1, and, in particular, the set

$$\mathcal{N} := \left\{ x \in \Omega : \sum_{i=1}^{h} w_i(x) = 0 \right\}$$

is a rectifiable set of codimension 1, made of the union of a finite number of $\mathcal{C}^{1,\alpha}$ smooth sub-manifolds (points if $\Omega \subset \mathbb{R}$, curves if $\Omega \subset \mathbb{R}^2$, and in general embedded surfaces if $\Omega \subset \mathbb{R}^N$). For any $n \in \mathbb{N}$, we introduce the renormalized solution

$$\bar{\mathbf{w}}_n := \left(\frac{w_{1,\beta_n}}{\|w_{1,\beta_n}\|_{L^\infty(\Omega)}}, \dots, \frac{w_{N,\beta_n}}{\|w_{N,\beta_n}\|_{L^\infty(\Omega)}}, u_{\beta_n} \right)$$

which is well defined since, by assumption, for any n and $1 \leq i \leq k$, $w_{i,n} > 0$. Let us observe that, since the first h components of $\mathbf{w}_n$ do not vanish as $\beta \to +\infty$, the corresponding first h components of $\bar{\mathbf{w}}_n$ are just a comparable scaling of their respective counterparts: in particular they converge in $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ to a renormalized limit which is zero on the set $\mathcal{N}$ and strictly positive otherwise.

Including the scaling in the system, it follows that $\bar{\mathbf{w}}_n$ is a solution to

$$\begin{cases} -d_i \Delta \bar{w}_{i,\beta_n} = \left(-\omega_i + k_i u_{\beta_n} - \beta \sum_{j \neq i} a_{ij} \|w_{j,\beta_n}\|_{L^\infty(\Omega)} w_{j,\beta_n} \right) \bar{w}_{i,\beta_n} \\ -D \Delta u_{\beta_n} = \left(\lambda - \mu u_{\beta_n} - \sum_{i=1}^N k_i w_{i,\beta_n} \right) u_{\beta_n} \\ \partial_\nu \bar{w}_{i,\beta_n} = \partial_\nu u_{\beta_n} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

We are mostly interested in the equations satisfied by the densities $\bar{w}_{i,\beta_n}$ for $h+1 \leq i \leq k$. The maximum principle implies that $u_n \leq \lambda/\mu$: testing the i-th equation by the density $\bar{w}_{i,\beta_n}$ itself and using its positivity, we find

$$\int_\Omega |\nabla \bar{w}_{i,\beta_n}|^2 \leq \frac{k_i}{d_i} \frac{\lambda}{\mu} |\Omega|$$

that is, since by definition $\|\bar{w}_{1,\beta_n}\|_{L^\infty(\Omega)} = 1$, $\bar{w}_{i,\beta_n}$ is uniformly bounded in $H^1(\Omega)$ and it admits a weak limit $\bar{w}_i \in H^1(\Omega)$: the compact embedding in $L^2(\Omega)$ and the boundedness in $L^\infty(\Omega)$ imply also that $\bar{w}_{i,\beta_n} \to \bar{w}_i$ strongly in $L^p(\Omega)$ for any $p < \infty$. Let us show that $\bar{w}_i$ is not trivial: for each $h+1 \leq i \leq k$ (the other components are non trivial by assumption) we have that

$$\begin{cases} -d_i \Delta \bar{w}_{i,\beta_n} \leq (-\omega_i + k_i u_{\beta_n}) \bar{w}_{i,\beta_n} \\ \partial_\nu \bar{w}_{i,\beta_n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $g_{i,n} \in H^1(\Omega)$ be a solution to

$$\begin{cases} -d_i \Delta \bar{g}_{i,n} + \omega_i g_{i,n} = k_i u_{\beta_n} \bar{w}_{i,\beta_n} \\ \partial_\nu \bar{g}_{i,\beta_n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard arguments we have that $0 \leq \bar{w}_{i,\beta_n} \leq g_{i,n}$ and that

$$\|g_{i,n}\|_{\mathcal{C}^{0,\alpha}(\Omega)} \leq C \|g_{i,n}\|_{W^{2,p}(\Omega)} \leq C \frac{k_i}{d_i} \frac{\lambda}{\mu} \|\bar{w}_{i,\beta_n}\|_{L^p(\Omega)}$$

for any $p < \infty$ and suitable C and $\alpha > 0$. As a result, using the order relationship between w_{i,β_n} and $g_{i,n}$, we have

$$1 = \|\bar{w}_{i,\beta_n}\|_{L^\infty(\Omega)} \leq C \frac{k_i}{d_i} \frac{\lambda}{\mu} \|\bar{w}_{i,\beta_n}\|_{L^p(\Omega)}$$

which implies in particular, taking the strong limit in $L^p(\Omega)$, that $\bar{\mathbf{w}}_n \to \bar{\mathbf{w}}$ has all of its components which are non trivial. We are now in a position to reach the desired contradiction. Let us consider the equation satisfied by $\bar{w}_{i,\beta}$ for $h+1 \leq i \leq k$: scaling back the first h densities, we have

$$\begin{cases} -d_i \Delta \bar{w}_{i,\beta_n} \leq \left(-\omega_i + k_i u_{\beta_n} - \beta_n \sum_{1 \leq j \leq h} w_{j,\beta_n} \right) \bar{w}_{i,\beta_n} \\ \partial_\nu \bar{w}_{i,\beta_n} = 0 & \text{on } \partial\Omega. \end{cases}$$

From the previous discussion, $\bar{w}_{i,\beta_n} \to \bar{w}_i$ in $L^2(\Omega)$ and the limit is non trivial. We let

$$c_i := \int_\Omega \bar{w}_i^2 > 0.$$

For any $\varepsilon > 0$ and $n \in \mathbb{N}$, we consider the sets

$$\Omega_{\varepsilon,n} := \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \varepsilon \text{ and } \inf_{m \geq n} \left(\sum_{i=1}^h w_{i,\beta_m}(x) \right) \geq \varepsilon \right\}$$

By the uniform convergence of $\mathbf{w}_{\beta_n}$ and the properties of its limit configuration we have that for each $\Omega_{\varepsilon,n}$ is closed and $\Omega_{\varepsilon_1,n_1} \subseteq \Omega_{\varepsilon_2,n_2}$ whenever $\varepsilon_1 > \varepsilon_2$ and $n_1 < n_2$ and, finally,

$$\bigcup_{\varepsilon > 0, n \in \mathbb{N}} \Omega_{\varepsilon,n} = \Omega \setminus \mathcal{N}$$

Since, as we have already recalled, $\mathcal{L}^N(\mathcal{N}) = 0$, it follows that for any $\delta > 0$, there exist $\bar{\varepsilon} > 0$ and $\bar{n} \in \mathbb{N}$ such that $\mathcal{L}^N(\Omega \triangle \Omega_{\varepsilon,n}) \leq \delta$ for $0 < \varepsilon < \bar{\varepsilon}$ and $n \geq \bar{n}$. By the absolute continuity of the Lebesgue integral and the uniform integrability of converging sequences, there exists $\bar{\delta} > 0$ (and consequently $\bar{\varepsilon}$ and $\bar{n}$) such that

$$\int_{\Omega_{\bar{\varepsilon},\bar{n}}} \bar{w}_{i,\beta_m}^2 \geq \frac{c_i}{2} > 0 \qquad \text{for } m \in \mathbb{N} \text{ sufficiently large.}$$

On the other hand, testing the equation in $\bar{w}_{i,\beta_m}$ by $\bar{w}_{i,\beta_m}$ itself, we obtain

$$\int_\Omega \left[d_i |\nabla \bar{w}_{i,\beta_m}|^2 + \omega_i \bar{w}_{i,\beta_m}^2 + \beta_m \left(\sum_{1 \leq j \leq h} w_{j,\beta_n} \right) w_{i,\beta_m}^2 \right] \leq k_i \int_\Omega u_{\beta_m} w_{i,\beta_m}^2 \leq k_i \frac{\lambda}{\mu} |\Omega|.$$

Since the terms of the left hand side are positive, we can localize the integral on the sets $\Omega_{\bar{\varepsilon},\bar{n}}$ and find

$$\beta_m \inf_{\Omega_{\bar{\varepsilon},\bar{n}}} \left(\sum_{1 \leq j \leq h} w_{j,\beta_n} \right) \int_{\Omega_{\bar{\varepsilon},\bar{n}}} w_{i,\beta_m}^2 \leq k_i \frac{\lambda}{\mu} |\Omega|$$

that is

$$0 < \frac{c_i}{2} < \int_{\Omega_{\bar{\varepsilon},\bar{n}}} \bar{w}_{i,\beta_m}^2 \leq \frac{1}{\beta_m \bar{\varepsilon}} \cdot k_i \frac{\lambda}{\mu} |\Omega|$$

a contradiction when β_m is sufficiently large, and this proves the first claim.

As of now, we have established that positive solutions must converge to one of the trivial solutions $(0, \dots, 0, 0)$ or $(0, \dots, 0, \lambda/\mu)$: to conclude the proof, we show that the can only converge to the latter. For this, we can adopt the same reasoning of Lemma 3.14: suppose that $\mathbf{w}_n \to (0, \dots, 0, 0)$ in $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ we have that for n large enough

$$\begin{cases} -d_i \Delta w_{i,\beta_n} = \left(-\omega_i + k_i u_{\beta_n} - \beta_n \sum_{j \neq i} w_{j,\beta_n} \right) w_{i,\beta_n} \leq -\frac{\omega_i}{2} w_{i,\beta_n} \\ \partial_\nu w_{i,\beta_n} = 0 & \text{on } \partial\Omega \end{cases}$$

which implies that each w_{i,β_n} must be identically zero, against our positivity assumption. □

Combing the previous results, we can show

Theorem 4.6. *Let $\delta > 0$ and let us consider, for any $N \geq 1$, the family of coefficients*

$$\delta < D, d_1, \dots, d_N, \omega_1, \dots, \omega_N, k_1, \dots, k_N < \frac{1}{\delta}.$$

For $\beta \geq 0$, let $\mathscr{S}$ be the set of solutions $(w_1, \dots, w_N, u) \in \mathcal{C}^{2,\alpha}(\Omega)$ to (4.1) with any number $N + 1 \geq 2$ of components, with coefficients as above. For any $(w_1, \dots, w_N, u) \in \mathscr{S}$ we associated

$$P(w_1, \dots, w_N, u) = \int_\Omega \sum_{i=1}^N w_i.$$

Then $\bar{N} \in \mathbb{N}$ for which we have two alternatives

- *either there exists $(\bar{w}_1, \dots, \bar{w}_{\bar{N}}, \bar{u}) \in \mathscr{S}$ such that*
$$P(\bar{w}_1, \dots, \bar{w}_{\bar{N}}, \bar{u}) = \max_{N \geq 1, (w_1, \dots, w_N, u) \in \mathscr{S}} P(w_1, \dots, w_N, u);$$
- *there exist a sequence $(\bar{w}_{1,n}, \dots, \bar{w}_{\bar{N},n}, \bar{u}_n) \in \mathscr{S}$ and functions $(w_1, \dots, w_N) \in Lip(\bar{\Omega})$ and $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ such that*
  - *$(\bar{w}_{1,n}, \dots, \bar{w}_{\bar{N},n}, \bar{u}_n)$ are solutions of (4.1) for $\beta_n \to +\infty$;*
  - *$(w_{1,\beta}, \dots, w_{N,\beta}) \to (w_1, \dots, w_N)$ in $\mathcal{C}^{0,\alpha} \cap H^1(\overline{\Omega})$, $u_\beta \to u$ in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$;*
  - *$(w_1, \dots, w_N, u)$ solves (4.2) and*
$$P(\bar{w}_1, \dots, \bar{w}_{\bar{N}}, \bar{u}) = \sup_{N \geq 1, (w_1, \dots, w_N, u) \in \mathscr{S}} P(w_1, \dots, w_N, u).$$

We stress the fact that we have posed no conditions on $\beta > 0$ and $\mu > 0$. The proof of this theorem in contained in the previous results.

4.1. **A case in which the optimal repartition has more than one pack.** Under some assumptions, we now show that the configuration that maximizes the total population of predators (that is, the solution in Theorem 4.6) contains more than one non trivial components of w_i. To show this result, we shall first consider a simplified version the system (3.1), that is

$$\begin{cases} -\Delta w_1 = (-\omega + ku - \beta w_2)\, w_1 \\ -\Delta w_2 = (-\omega + ku - \beta w_1)\, w_2 \\ -\Delta u = (\lambda - kw_1 - kw_2)\, u \\ \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We observe that we are considering here a system with indistinguishable densities of predators (all the characterizing parameters in the equations are independent of the densities) and, more importantly, that the parameter $\mu = 0$: an extension of the result for $\mu > 0$ will be presented later. We recall (see Proposition 2.4 and Definition 2.6) that (4.4) has as simple solutions (that is, non trivial solutions with only one non trivial component of $(w_1, \dots, w_N)$) only the constant solution

$$(W, U) = \left(\frac{\lambda}{k}, \frac{\omega}{k}\right) \implies \int_\Omega W = \frac{\lambda}{k}|\Omega|, \int_\Omega U = \frac{\omega}{k}|\Omega|$$

Under this assumptions, we have

Lemma 4.7. *Let (w_1, w_2, u) be any solution of (4.4) with $\beta > 0$ with all non trivial components. Then*

$$\int_\Omega \sum_i^2 w_i = \frac{\lambda}{k}|\Omega| + \frac{1}{k}\int |\nabla \log u|^2 > \frac{\lambda}{k}|\Omega|$$

and

$$\int_\Omega u = \frac{\omega}{k}|\Omega| + \frac{\beta}{\lambda}\int w_1 w_2 + \frac{\omega}{k\lambda}\int |\nabla \log u|^2 > \frac{\omega}{k}|\Omega|.$$

Remark 4.8. Equivalently, we could have compared the solutions of (4.4) with $\beta > 0$ with any non trivial solution in the case $\beta = 0$ (see Lemma 3.3).

Proof. We recall that, thanks to Lemma 3.6, if u is constant, so are all the other components, and thus in this case the solution has to be simple. As a result, we can assume that u is not constant: the existence of such solutions is already known thanks to Theorem 3.8 and 3.11. Let us consider the equation in u. By the maximum principle, $u > 0$ and thus we can divide the two members by u and integrate over Ω, obtaining

$$\int_\Omega \sum_i^2 w_i = \frac{1}{k}\int \left(\lambda + \frac{\Delta u}{u}\right) = \frac{\lambda}{k}|\Omega| + \frac{1}{k}\int |\nabla \log u|^2 > \frac{\lambda}{k}|\Omega| \tag{4.5}$$

where the strict inequality follows by the fact that u is not a constant. Similarly, integrating directly the equations of the system, summing them and using the previous identity, we obtain

$$\int_\Omega u = \frac{\omega}{k}|\Omega| + \frac{\beta}{\lambda}\int w_1 w_2 + \frac{\omega}{k\lambda}\int |\nabla \log u|^2 > \frac{\omega}{k}|\Omega|$$

and this concludes the proof. □

As a result, according to the model (4.4), competition ($\beta > 0$) is always advantageous both for the predators and for the preys, in the sense that the total population of predators (and preys) is greater in the case of two groups computing for the same territory, than in the case of only one group. We now wish to extend this result in the more realistic case of model (3.1), that is

$$\begin{cases} -\Delta w_1 = (-\omega + ku - \beta w_2)\, w_1 \\ -\Delta w_2 = (-\omega + ku - \beta w_1)\, w_2 \\ -\Delta u = (\lambda - \mu u - kw_1 - kw_2)\, u \\ \partial_\nu w_i = \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

when $\mu > 0$. To do this, let us first observe that the same computations as those in the previous result yield the identity

$$\begin{aligned}\int_\Omega \sum_i^2 w_i &= \frac{\lambda}{k}|\Omega| - \frac{\mu}{k}\int u + \frac{1}{k}\int |\nabla \log u|^2 \\ &= \frac{\lambda k - \mu\omega}{k^2}|\Omega| + \frac{\mu}{k}\left(\frac{\omega}{k}|\Omega| - \int u\right) + \frac{1}{k}\int |\nabla \log u|^2\end{aligned}$$

which, by uniform convergence of the densities as $\beta \to +\infty$, is valid also in the limit case of segregation. Unlike the case $\mu = 0$, a direct comparison of previous formula with the case of only one population of predators is non immediate, since in general we can show that the second term in the last expression is negative

$$\int u > \frac{\omega}{k}|\Omega|.$$

As a result, we need to carefully estimate the various contributions on the identity, in order to show that, when μ is sufficiently small, the total population of predators increases in the case of two non trivial components w_1 and w_2.

We want to stress that this is not a trivial task. Indeed, an a priori estimate on the solutions which is independent of μ may not be true, for several reasons:

- from the equation in u, we can only say a priori that $u \leq \lambda/\mu$. If $\mu \to 0$, we have no reason to conclude that the solutions of (4.6) converge to solutions of (4.4);
- one may wonder whether the previous bound is not sharp and that is may be attained only by "spurious" solutions such as $(0, 0, \lambda/\mu)$. But this assertion is not in general true, and to see this we can recall that, by Theorem 3.8, non constant (thus "genuine") solutions bifurcate from
$$\left(\frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}, \frac{\lambda k - \mu\omega}{\mu\beta + 2k^2}, \frac{\lambda\beta + 2k\omega}{\mu\beta + 2k^2}\right) \qquad \text{for } \beta\frac{\lambda k - \mu\omega}{\mu\beta + 2k^2} = \gamma_n$$
where γ_n is the n-th eigenvalue of the Laplace operator with Neumann boundary conditions. For μ sufficiently small and γ_n large (and, consequently, β large), we have non constant solutions for which u is close (at least) in the uniform topology to the upper bound λ/μ.

We thus focus on the one-dimensional case, for which (see Theorem 3.12) we have already established the existence of segregated solutions and pointed out their symmetries (Remark 3.10). As a result, for $\Omega = (-a, a)$, $a > 0$ and $\mu > 0$ sufficiently small, we have a continuum of solutions such that $w_1 - w_2$ vanishes only for $x = 0$. Sending the competition parameter to infinity $\beta \to +\infty$, by Lemma 3.16 we can start by considering the system

$$\begin{cases} -w'' = (-\omega + ku)\, w \\ -u'' = (\lambda - \mu u - kw)\, u \\ w(0) = w'(a) = u'(0) = u'(a) = 0 \end{cases} \quad \text{in } (0, a) \tag{4.7}$$

for $\mu > 0$, for which the identity (4.5) reduces to

$$\int_0^a w = \frac{\lambda}{k}a - \frac{\mu}{k}\int_0^a u + \frac{1}{k}\int_0^a |(\log u)'|^2$$

Proposition 4.9. *Let (w, u) be any classical solution of (4.7) with both components non negative and nontrivial. For $\mu > 0$ sufficiently small*

$$\int_0^a w > \frac{\lambda}{k}a.$$

Let us observe that, since the solutions of (4.6) converge for $\beta \to +\infty$ to segregated solutions, the previous result implies an improvement of Theorem 4.6, and in particular we have

Theorem 4.10. *Under the assumptions of Theorem 4.6, let us assume moreover that the coefficients in (4.1) do not depend on the index i and that $\Omega \Subset \mathbb{R}$. If $\mu > 0$ sufficiently small the solution of (4.1) that* *maximizes*

$$P(w_1, w_2, \dots, w_N, u) = \int_\Omega \sum_{i=1}^N w_i$$

has at least $N \geq 2$ *non trivial components and* $\beta > 0$.

We divide the proof of 4.9 in two separate results. Letting all the parameters in (4.7) fixed a part from $\mu > 0$, we have

Lemma 4.11. *Let* (w, u) *be any classical solution of* (4.7) *with both components non negative and nontrivial. For any* $\varepsilon > 0$ *there exists* $\bar{\mu} > 0$ *such that*

$$\mu \int_0^a u \leq \varepsilon \qquad \text{if } \mu \in (0, \bar{\mu}).$$

Proof. Let (w_n, u_n) be a sequence of positive solutions to (4.7) for $\mu = \mu_n$, and let us assume, by contradiction, that

$$\mu_n \int_0^a u_n > C > 0.$$

First of all, the maximum principle, as already observed, implies that $u_n \leq \lambda/\mu_n$, that is

$$\mu_n \int_0^a u_n \leq a\lambda$$

so we can assume that, up to a subsequence

$$\lim_{n \to +\infty} \mu_n \int_0^a u_n = aC \in (0, a\lambda]$$

for some constant $C > 0$. We can introduce the scaled functions $(\bar{w}_n, \bar{u}_n)$ as

$$\bar{w}_n := \left(\int_0^a (w_n')^2 \right)^{-1/2} w_n, \quad \bar{u}_n := \left(\frac{1}{a} \int_0^a u_n \right)^{-1} u_n,$$

which are solutions to

$$\text{(4.8)} \qquad \begin{cases} -\bar{w}_n'' = (-\omega + k_n \bar{u}_n) \bar{w}_n \\ -\bar{u}_n'' = (\lambda - \mu_n \bar{u}_n - k_n' \bar{w}_n) \bar{u}_n & \text{in } (0, a) \\ \bar{w}_n(0) = \bar{w}_n'(a) = \bar{u}_n'(0) = \bar{u}_n'(a) = 0 \end{cases}$$

where for convenience we let

$$k_n := k \frac{1}{a} \int_0^a u_n, \quad \mu_n := \mu \frac{1}{a} \int_0^a u_n, \quad k_n' := k \left(\int_0^a (w_n')^2 \right)^{1/2}.$$

We observe that, from the assumption of contradiction,

$$k_n \to +\infty \qquad \text{and} \quad \mu_n \to C \in (0, \lambda),$$

while we have no information on k_n'. Moreover, by definition and the Dirichlet boundary condition at zero, the sequence $\{\bar{w}_n\}_{n \in \mathbb{N}}$ is bounded in $H^1(0, a)$, and by positivity, also $\{\bar{u}_n\}_{n \in \mathbb{N}}$ is bounded in $H^1(0, a)$, indeed by testing the equation in $\bar{u}_n$ with $\bar{u}_n$ itself we obtain

$$\int_0^a (\bar{u}_n')^2 + \mu_n \bar{u}_n^3 \leq \lambda \int_0^a \bar{u}_n^2$$

and the claim follows from the assumption $\mu_n \to C > 0$. By the embedding theorems we have that, up to a subsequence, both $\{\bar{w}_n\}_{n \in \mathbb{N}}$ and $\{\bar{u}_n\}_{n \in \mathbb{N}}$ converge uniformly in $(0, a)$ to their respective weak $H^1(0, a)$ limits, $\bar{w}_\infty$ and $\bar{u}_\infty$. Moreover, by renormalization and strong convergence, we have

$$\int_0^a \bar{u}_\infty = a$$

and thus, in particular, $\bar{u}_\infty$ is non trivial. Finally, from the equation in $\bar{w}_n$ we see that

$$k_n \int_0^a \bar{u}_n \bar{w}_n^2 = \int_0^a (\bar{w}_n')^2 + \omega \bar{w}_n^2 \leq C'.$$

Since $k_n \to +\infty$, by the uniform convergence we have that

$$\bar{u}_n \bar{w}_n \to \bar{u}_\infty \bar{w}_\infty \equiv 0 \qquad \text{uniformly in } (0,a).$$

Step 1) We now proceed and exclude the possibility that the sequence k'_n is bounded. By the uniform convergence we have

$$k'_n \bar{u}_n \bar{w}_n \to 0 \qquad \text{uniformly in } (0,a).$$

It follows, passing to the limit in the equation satisfied by $\bar{u}_n$, that $\bar{u}_\infty$ is a non trivial solution of

$$\begin{cases} -\bar{u}''_\infty = (\lambda - C\bar{u}_\infty)\,\bar{u}_\infty & \text{in } (0,a) \\ \bar{u}'_\infty(0) = \bar{u}'_\infty(a) = 0. \end{cases} \tag{4.9}$$

From Lemma 3.15, $\bar{u}_\infty$ can thus be only the constant λ/C, and finally, thanks to renormalization, $C = \lambda$ and $\bar{u}_\infty \equiv 1$. By the uniform convergence of $\bar{u}_n$ to its limit, we have that for n sufficiently large

$$\bar{u}_n > \frac{1}{2} \qquad \text{in } (0,a).$$

Using this estimate in the equation for $\bar{w}_n$ we have

$$\begin{cases} -\bar{w}''_n = (-\omega + k_n \bar{u}_n)\,\bar{w}_n > (-\omega + k_n/2)\,\bar{w}_n \\ \bar{w}_n(0) = \bar{w}'_n(a) = 0. \end{cases}$$

Since $k_n \to +\infty$, as soon as

$$(-\omega + k_n/2) > \left(\frac{\pi}{2a}\right)^2$$

(the principle eigenvalue of the equation), by the comparison principle we have $\bar{w}_n \equiv 0$, against the assumption that the functions (w_n, u_n) are positive in $(0,a)$.

Step 2) As a result, we have that $k'_n \to +\infty$: similarly we can show that $k'_n \geq C k_n$. Indeed, let us assume by contradiction that $k'_n/k_n \to 0$, testing the equation in $\bar{w}_n$ with φ smooth and compactly supported, we find

$$k_n \int_0^a \bar{u}_n \bar{w}_n \varphi = \int_a^a (\bar{w}'_n \varphi + \omega \bar{w}_n \varphi)$$

and the right-hand side is bounded in n, so that

$$k'_n \int_0^a \bar{u}_n \bar{w}_n \varphi = \frac{k'_n}{k_n} k_n \int_0^a \bar{u}_n \bar{w}_n \varphi \to 0$$

for all test function φ. It follows that the weak and uniform limit of $\bar{u}_n$, $\bar{u}_\infty$ solves again (4.9), and thus

$$\bar{u}_n \to 1 \qquad \text{uniformly in } (0,a).$$

We can then reach a contradiction as in Step 1).

Step 3) We now show that $k_n \geq C k'_n$. Again by contradiction, let us assume that $k_n/k'_n \to 0$: integrating the equation in $\bar{u}_n$ we obtain

$$k'_n \int_0^a \bar{u}_n \bar{w}_n = \int_0^a \bar{u}_n (\lambda - \mu_n \bar{u}_n)$$

and the right-hand side is bounded uniformly in n. It follows that

$$k_n \int_0^a \bar{u}_n \bar{w}_n = \frac{k_n}{k'_n} k'_n \int_0^a \bar{u}_n \bar{w}_n \to 0.$$

But then, testing the equation in $\bar{w}_n$ with $\bar{w}_n$ itself, we obtain, thanks to the renormalization of $\bar{w}_n$ and the uniform convergence,

$$0 < C' < \int_0^a (\bar{w}'_n)^2 + \omega \bar{w}_n^2 = k_n \int_0^a \bar{u}_n \bar{w}_n^2 \leq \|\bar{w}_n\|_{L^\infty} \cdot k_n \int_0^a \bar{u}_n \bar{w}_n \to 0$$

a contradiction.

Step 4) In summary, we have shown so far that

$$k_n \to +\infty \qquad \text{and} \quad k'_n = O(k_n).$$

We already know that, up to a subsequence, the sequence $\{\bar{w}_n\}_{n\in\mathbb{N}}$ converges to continuous function $\bar{w}_\infty$: we show that $\bar{w}_\infty \equiv 0$. Let us assume that this is not the case, then there exist $0 \le x_0 < x_1 \le a$ be such that

$$\inf_{x\in[x_0,x_1]} \bar{w}_n > C > 0 \qquad \text{for all } n \text{ sufficiently large}$$

Then for some positive constants C', C'' and n sufficiently large, we have

$$\begin{cases} -\bar{u}_n'' = (\lambda - \mu_n \bar{u}_n - k_n' \bar{w}_n)\, \bar{u}_n < -C'' k_n' \bar{u}_n & \text{in } (x_0, x_1) \\ \bar{u}_n < C' & \text{in } (0, a). \end{cases}$$

By comparison with the super-solution

$$x \mapsto C' \cosh\left[(C'' k_n')^{1/2}\left(x - \frac{x_0 + x_1}{2}\right)\right] \Big/ \cosh\left[(C'' k_n')^{1/2}\left(\frac{x_1 - x_0}{2}\right)\right]$$

we have that, for $\varepsilon > 0$ small

$$\sup_{x\in[x_0+\varepsilon,x_1-\varepsilon]} \bar{u}_n \le C' \cosh\left[(C'' k_n')^{1/2}\left(\frac{x_1 - x_0}{2} - \varepsilon\right)\right] \Big/ \cosh\left[(C'' k_n')^{1/2}\left(\frac{x_1 - x_0}{2}\right)\right].$$

Coming back to the equation in $\bar{w}_n$, and recalling that $k_n = O(k_n')$, we can pass to the limit and obtain

$$-\bar{w}_\infty'' = -\omega \bar{w}_\infty \qquad \text{in } [x_0 + \varepsilon, x_1 - \varepsilon].$$

We now observe that the previous reason is true for any ε and any interval of positivity $[x_0, x_1]$ of $\bar{w}_\infty$. As a result, in any interval of positivity $[x_0, x_1] \subset [0, a]$ we have

$$\begin{cases} -\bar{w}_\infty'' = -\omega \bar{w}_\infty \\ \bar{w}_\infty(x_0) = \bar{w}_\infty(x_1) = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} -\bar{w}_\infty'' = -\omega \bar{w}_\infty \\ \bar{w}_\infty(x_0) = \bar{w}_\infty'(x_1) = 0 \end{cases}$$

In both case, reasoning as in Step 1), we see that $\bar{w}_\infty \equiv 0$ in $[x_0, x_1]$, meaning that there are no intervals in $(0, a)$ where $\bar{w}_\infty$ is positive, and thus

$$\bar{w}_n \to 0 \qquad \text{uniformly in } (0, a).$$

Now we can repeat the reasoning of Step 3): integrating the equation in $\bar{u}_n$ we obtain

$$k_n \int_0^a \bar{u}_n \bar{w}_n \sim k_n' \int_0^a \bar{u}_n \bar{w}_n = \int_0^a \bar{u}_n (\lambda - \mu_n \bar{u}_n)$$

while, from the equation in $\bar{w}_n$ and the uniform limit proved before

$$0 < C' < \int_0^a (\bar{w}_n')^2 + \omega \bar{w}_n^2 = k_n \int_0^a \bar{u}_n \bar{w}_n^2 \le \|\bar{w}_n\|_{L^\infty} \cdot k_n \int_0^a \bar{u}_n \bar{w}_n \to 0$$

a contradiction. □

Lemma 4.12. *Let (w, u) be any classical solution of (4.7) with both components non negative and nontrivial. There exist two constants $C > 0$ and $\bar{\mu} > 0$ such that*

$$\int_0^a |(\log u)'|^2 > C \qquad \text{if } \mu \in (0, \bar{\mu}).$$

Proof. Let us consequence a (w_n, u_n) of positive solutions to (4.7) for $\mu = \mu_n$ such that

$$\lim_{n\to+\infty} \int_0^a |(\log u_n)'|^2 = 0$$

By the embeddings theorems we have that

$$\lim_{n\to+\infty} \frac{\sup_{x,y\in(0,a)} |u_n(x) - u_n(y)|}{\|u_n\|_{L^\infty(0,a)}} = 0.$$

and, moreover, since u_n is positive,

$$\lim_{n\to+\infty} \frac{\inf_{(0,a)} u_n}{\sup_{(0,a)} u_n} = 1. \tag{4.10}$$

From the equation in w_n we see that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded from above and away from zero. Indeed, if $\inf u_n \to +\infty$, then for n sufficiently large

$$\begin{cases} -w_n'' = (-\omega + ku_n)\, w_n > \left(\frac{\pi}{2a}\right)^2 w_n \\ w_n(0) = w_n'(a) = 0 \end{cases} \qquad \Longrightarrow \; w_n \equiv 0$$

while, if $\sup u_n \to 0$, then for n sufficiently large

$$\begin{cases} -w_n'' = (-\omega + ku_n)\, w_n < \left(\frac{\pi}{2a}\right)^2 w_n \\ w_n(0) = w_n'(a) = 0 \end{cases} \qquad \Longrightarrow \; w_n \equiv 0$$

and in both cases we reach a contradiction with the positivity of w_n. This, together with (4.10) implies at u_n converges uniformly to a nontrivial constant C. Again by the equation in w_n, we have that necessarily

$$u_n \to \frac{\omega}{k} + \frac{1}{k}\left(\frac{\pi}{2a}\right)^2 \qquad \text{uniformly in } (0, a).$$

Up to a renormalization, we also deduce that

$$\left(\|w_n\|_{L^\infty(0,a)}\right)^{-1} w_n \to \sin\left(\frac{\pi}{2a}x\right)$$

strongly in $H^1(0, a)$ and uniformly. Integrating the equation in u_n, we have

$$k\int_0^a u_n w_n = \lambda \int_0^a u_n - \mu_n \int_0^a u_n^2$$

and thus we have that $\{w_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $(0, a)$, so that

$$w_n \to C \sin\left(\frac{\pi}{2a}x\right)$$

for a non negative constant C. Consequently, using these information in the equation satisfied by u_n, we have that u_n is bounded uniformly in $H^1(0, a)$, and in we can thus take the weak limit of the equation to see that

$$0 = \left[\lambda - kC\sin\left(\frac{\pi}{2a}x\right)\right]\left[\frac{\omega}{k} + \frac{1}{k}\left(\frac{\pi}{2a}\right)^2\right],$$

which is impossible to solve and gives us the desired contradiction. □

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