OPTIMAL REGULARITY RESULTS RELATED TO A PARTITION PROBLEM INVOLVING THE HALF-LAPLACIAN

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Abstract. For a class of optimal partition problems involving the half-Laplacian operator and a subcritical cost functionals, we derive the optimal regularity of the density-functions which characterize the partitions, for the entire set of minimizers. We present a numerical scheme based on the arguments of the proof and we collect some numerical results related to the problem.

1. Introduction

In recent time, the study of nonlocal operators has become a dominant subject in the regularity theory of minimization problems and elliptic equations. Originally inspired by modelling reasons, the study of non-local diffusion operators has revealed important in order both to test and to extend already understood theories concerning the behaviour of solutions to local problems.

Of the many non-local operators now object of study in the literature, this paper is concerned with possibly the easiest yet most fundamental one: the half-Laplacian. Given a smooth function $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, the half-Laplacian operator $(-\Delta)^{1/2}$ is defined as the singular integral

$$(-\Delta)^{1/2}u := C_N \,\mathrm{pv} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+1}} \mathrm{d}y$$

where the constant C_N is a normalization constant and pv stands for the principal value. For non-smooth functions, whenever possible, the operator is defined in the distributional sense (see [9], or the more recent [8], for comprehensive theory of the operator). As it is now well known, the above operator is related both to the infinitesimal generator of a Levy α-stable diffusion process and, via the Fourier transform $\mathcal{F}$, to the multiplication operator whose symbol is given by $|\xi|$ (see [8, Proposition 3.3]), that is

$$\forall u \in \mathcal{S}(\mathbb{R}^k) \quad (-\Delta)^{1/2}u = \mathcal{F}^{-1}(|\xi|\hat{u})$$

where $\mathcal{S}(\mathbb{R}^k)$ is the space of Schwartz functions, $\hat{u} = \mathcal{F}(u)$ and $\mathcal{F}^{-1}$ is the inverse transform. Moreover, from a variational point of view, the half-Laplacian can be

1991 *Mathematics Subject Classification.* Primary: 49Q10; secondary: 35B40 35R11 45C05 81Q05 82B10.

Key words and phrases. Square root of the Laplacian, spatial segregation, strongly competing systems, optimal regularity of limiting profiles, singular perturbations.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 321186 : "Reaction-Diffusion Equations, Propagation and Modelling" held by Henri Berestycki, and under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 339958 : "Complex Patterns for Strongly Interacting Dynamical Systems" held by Susanna Terracini.

related to the differential of the fractional Sobolev seminorm of $H^{1/2}(\mathbb{R}^N)$, that is

$$|u|^2_{H^{1/2}(\mathbb{R}^N)} := \langle (-\Delta)^{1/2} u, u\rangle = \frac{C_N}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}}\mathrm{d}x\mathrm{d}y.$$

The paper is devoted to the study of the regularity of optimal partition problems involving the fractional operator $(-\Delta)^{1/2}$. With this we mean that, given a set $\Omega \subset \mathbb{R}^N$ and a cost functional J associated to a suitable set of partitions of Ω, we wish to find the regularity shared by *all* the partitions that minimize J. More precisely, let us consider the functional space

$$H^{1/2}(\mathbb{R}^N) := \left\{u : \|u\|^2_{H^{1/2}(\mathbb{R}^N)} := |u|^2_{H^{1/2}(\mathbb{R}^N)} + |u|^2_{L^2(\mathbb{R}^N)} < +\infty\right\}$$

and let $\Omega \subset \mathbb{R}^N$ be bounded and smooth set (i.e., with at least $\mathcal{C}^1$ boundary). Given some suitable functions $F_i \colon \Omega\times\mathbb{R}\to\mathbb{R}$, we introduce the functional

$$J(u_1,\dots,u_k) := \begin{cases} \displaystyle\sum_{i=1}^k \left(\frac{1}{2}|u_i|^2_{H^{1/2}(\mathbb{R}^N)} + \int_\Omega F_i(x,u_i)\mathrm{d}x\right) & \\ & \text{if } u_i\cdot u_j = 0 \text{ a.e. for every } j\neq i \\ +\infty & \text{otherwise} \end{cases} \tag{1.1a}$$

and set the optimal partition problem on

$$\mathbb{S}^k_{L^2} := \left\{(u_1,\dots,u_k) : u_i \in H^{1/2}_\Omega(\mathbb{R}^N), \|u_i\|_{L^2(\mathbb{R}^N)} = 1\right\} \tag{1.1b}$$

where we used the notation $H^{1/2}_\Omega(\mathbb{R}^N) := \{w \in H^{1/2}(\mathbb{R}^N) : w|_{\mathbb{R}^N\setminus\Omega} = 0\}$. The main results we shall prove in the paper are the followings.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N$ be bounded and smooth set. For each $i = 1,\dots,k$, let $F_i\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function (that is, $(x,s)\mapsto F_i(x,s)$ is measurable in x and continuous in s) such that*

$$|F_i(x,s)| \le C_i(1+|s|^p) \qquad \forall x\in\Omega, s\in\mathbb{R}$$

for a suitable constant $C_i \ge 0$, where $p < p^\sharp = \frac{2N}{N-1}$. Then there exists at least a minimizer of J in $\mathbb{S}^k_{L^2}$. Moreover, if $F_i(x,\cdot)\in\mathcal{C}^1(\mathbb{R})$ for a.e. $x\in\Omega$, then any minimizer $\mathbf{u} := (u_1,\dots,u_k)\in\mathcal{C}^{0,\alpha}(\mathbb{R}^N;\mathbb{R}^k)$ for any $\alpha\in(0,1/2)$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let us assume also that*

(A) *each function F_i is independent of x, $F_i\in\mathcal{C}^{2,\varepsilon}(\mathbb{R})$ for some $\varepsilon>0$ and $F_i'(0) = 0$.*

Then any minimizer $\mathbf{u}$ of J over the set $\mathbb{S}^k_{L^2}$ belongs to $\mathcal{C}^{0,1/2}(\mathbb{R}^N;\mathbb{R}^k)$ and satisfies the following Euler-Lagrange equation

$$u_i\left((-\Delta)^{1/2}u_i - F_i'(u_i)\right) = 0 \qquad \text{a.e. } in\ \Omega.$$

Remark 1.3. One could also consider partition problems of unbounded domains, for example with $\Omega = \mathbb{R}^N$, if the functions F_i can be used to ensure compactness: this can be achieved, for instance, if $F_i(x,s) = V(x)s^2$, with V positive and $V(x)\to+\infty$ for $|x|\to\infty$. In such a case the correct functional setting in given by the space

$$H^{1/2}_V(\mathbb{R}^N) := \left\{w\in H^{1/2}(\mathbb{R}^N) : \|w\|^2_{H^{1/2}_V(\mathbb{R}^N)} := |w|^2_{H^{1/2}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} Vw^2\mathrm{d}x < \infty\right\}.$$

Associating to the bounded set Ω its indicator function

$$\chi_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega, \end{cases}$$

we see that $H^{1/2}_\Omega(\mathbb{R}^N) \equiv H^{1/2}_{\chi_\Omega}(\mathbb{R}^N)$. We shall not address this extension in the following, though the theory here developed may be used also to cover this case with little modifications.

At the moment no result asserting the regularity of the partition sets $\{(\omega_1, \dots, \omega_k)\}$ is known. In any case we observe that from Theorem 1.1 we can deduce that any subset $\omega_i = \{u_i > 0\} \cup \{u_i < 0\}$ is an open set, which is already a non trivial result. Theorems 1.1 and 1.2 are analogous to well established results found in the case of standard diffusion operators, see for instance [3, 10]: in particular, they constitute the first step in the proof of the regularity of the free-boundary $\cup_i \partial\omega_i$, as done in [3, 11].

As a possible application, we can consider the case $F_i \equiv 0$. In such a situation, the optimal partition problem (1.1) is precisely given by the problem of finding k disjoint subsets $\omega_1, \dots, \omega_k$ of Ω such that the functional

$$(\omega_1, \dots, \omega_k) \mapsto \sum_{i=1}^k \lambda_1(\omega_i)$$

is minimal. Here $\lambda_1(\omega_i)$ stands for the first eigenvalue of the half-Laplacian in ω_i, defined as

$$\lambda_1(\omega_i) := \inf \left\{ |u|^2_{H^{1/2}(\mathbb{R}^N)} : \|u\|_{L^2(\mathbb{R}^N)} = 1,\, u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \omega_i \right\}.$$

Remark 1.4. From a point of view of the applications, mainly linked to pattern formation in relativistic quantum systems, one could also consider a slightly different formulation of the optimal partition problem, as follows. Let us fix $k \in \mathbb{N}$ non-negative constants $m_1, \dots, m_k$ and let us introduce the operator $(-\Delta + m_i^2)^{1/2}$, which acts on smooth functions as

$$\forall u \in \mathcal{S}(\mathbb{R}^k) \quad (-\Delta + m_i^2)^{1/2} u = \mathcal{F}^{-1}((\xi^2 + m_i^2)^{1/2} \hat{u}).$$

Accordingly, one could introduce as a cost functional

$$R(u_1, \dots, u_k) := \begin{cases} \displaystyle\sum_{i=1}^k \left(\frac{1}{2} \langle (-\Delta + m_i^2)^{1/2} u_i, u_i \rangle + \int_\Omega F_i(x, u_i) \mathrm{d}x \right) \\ \qquad\qquad\qquad \text{if } u_i \cdot u_j = 0 \text{ a.e. for every } j \neq i \\ +\infty \qquad\qquad\quad \text{otherwise} \end{cases}$$

again defined over the set $\mathbb{S}^k_{L^2}$. The same regularity results available for the functional J can be recast and extended without effort to the case of the functional R.

The last section is devoted to some numerical results. The simulations are obtained using an approximation scheme which is based on the proof of the Theorems 1.1 and 1.2: some comparisons with the results obtained in the case of the standard Laplacian are also presented.

To conclude, we would like to mention that results similar to those discussed above are available also in the case of any fractional power of the Laplacian $(-\Delta)^s$ with $s \in (0, 1)$: some of the needed preliminaries can be already found in [13].

2. Proof of the results

As a first step, we shall prove that, under the assumptions of Theorem 1.1, the optimal partition problem admits at least a solution. Later we shall concentrate on the regularity of the whole set of solutions.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded and smooth set. For each $i = 1, \dots, k$, let $F_i \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, $(x,s) \mapsto F_i(x,s)$ is measurable in x and continuous in s) such that*

$$|F_i(x,s)| \le C_i(1+|s|^p) \qquad \forall x \in \Omega, s \in \mathbb{R}$$

for a suitable constant $C_i \ge 0$, where $p < p^\sharp = \frac{2N}{N-1}$. Then there exists a minimizer of J in $\mathbb{S}^k_{L^2}$.

Proof. The lemma follows by the direct method of the calculus of variations. Indeed, we evince directly from the assumptions on the functions F_i that the functional J is weakly lower semicontinuous in $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$ and moreover, since

$$\lim_{\|\mathbf{u}\|_{H^{1/2}} \to \infty} J(\mathbf{u}) = +\infty,$$

the functional J is also coercive in the weak topology of $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$ (see [7, Example 1.14]). □

The regularity of the solutions to the previous minimization problem is in general hard to study directly. In order to simplify the analysis, in what follows we shall introduce two families of functionals which are related to the previous one in a precise way. The first family precisely implements the disjointness constraint in a relaxed way, through a penalization term: in particular we shall show that any sequence of minima to the family of functional converges to a minimum of the original functional. Our goal is to show that the topology of this convergence is sufficiently strong in order to ensure the regularity of the limiting densities. Unfortunately, since no result is known about the uniqueness of the optimal partition, the first proposed approximating procedure may fail to conclude the regularity of the whole set of optimal partitions. To avoid this issue, we need to introduce another family of functionals.

We start with the easier family of functionals.

Definition 2.2. Under the functional setting of Theorem 1.1, for any $\beta > 0$, let us introduce

$$J_\beta(u_1, \dots, u_k) := \sum_{i=1}^k \left(\frac{1}{2}|u_i|^2_{H^{1/2}(\mathbb{R}^N)} + \int_\Omega F_i(x,u_i)\mathrm{d}x \right) + \beta \sum_{j<i} \int_\Omega u_i^2 u_j^2 \mathrm{d}x.$$

Lemma 2.3. *Under the assumptions of Theorem 1.1, for every $\beta > 0$ there exists a minimizer $\mathbf{u}_\beta \in \mathbb{S}^k_{L^2}$ of J_β. Moreover, there exists a constant $C > 0$ (independent of β) such that $\|\mathbf{u}_\beta\|_{H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)} \le C$.*

Proof. The proof is analogous to the one given in the limiting case $\beta = +\infty$. The main difference is represented by the presence of the interaction term, which is not sub-critical if $N \ge 3$. In this situation, it is sufficient to recall that, thanks positivity of β, the last term is lower semicontinuous, as a consequence of the Fatou's Lemma. □

Lemma 2.4. *It holds $\Gamma - \lim_{\beta\to+\infty} J_\beta = J$ (w.r.t. the weak $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$-topology). Moreover, any sequence of minimizers $\{\mathbf{u}_\beta\}$ to J_β converges weakly in $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$, up to a subsequence, to a minimizer of J.*

Proof. The family of functionals J_β is increasing in β and converges pointwise to the functional J. As a consequence $\Gamma - \lim J_\beta = J$. The family J_β is also equi-coercive and this implies, up to subsequences, the convergence of the minimizers. See [7, Proposition 5.4] and [7, Corollary 7.20] for further details. □

As mentioned before, even though the family $\{\mathbf{u}_\beta\}$ converges, up to subsequences, to a minimizer of J, at the moment we can not say that any minimizer of J can be approximated in this way. In order to obtain a stronger conclusion, we need another step, involving the introduction of another functional, which will be the main object of the analysis in the following. For this purpose, let

$$e(s) := \sqrt{1+s^2}$$

(we observe preliminarily that $|e'(s)| < 1$ for any $s \in \mathbb{R}$) and let $\bar{\mathbf{u}} \in \mathbb{S}^k_{L^2}$ be any minimizer of J.

Definition 2.5. Under the functional setting of Theorem 1.1, for any $\beta > 0$, we let

$$J^*_\beta(u_1,\dots,u_k) := \sum_{i=1}^k \left(\frac{1}{2}|u_i|^2_{H^{1/2}(\mathbb{R}^N)} + \int_\Omega \left[F_i(x,u_i) + e(u_i - \bar{u}_i)\right] \mathrm{d}x \right) + \frac{\beta}{2} \sum_{j<i} \int_\Omega u_i^2 u_j^2 \mathrm{d}x.$$

It is immediate to see that the proof of existence of minimizers developed for the functional J_β covers also the functional J^*_β. Moreover, since the functional J^*_β can be decomposed as

$$J^*_\beta(\mathbf{u}) = J_\beta(\mathbf{u}) + \sum_{i=1}^k \int_\Omega e(u_i - \bar{u}_i)\mathrm{d}x$$

it easily follows that any the sequence of minima $\{\mathbf{u}_\beta\}$ convergence weakly in $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$ and strongly in $L^2(\mathbb{R}^N;\mathbb{R}^k)$ to the minimum $\bar{\mathbf{u}}$ of J.

Lemma 2.6. *There exists $C > 0$ independent of β such that*

$$\|\mathbf{u}_\beta\|^2_{H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)} + \beta \int_{\mathbb{R}^N} \sum_{j\neq i} u_{i,\beta}^2 u_{j,\beta}^2 \mathrm{d}x \le C.$$

Moreover if $F_i(x,\cdot) \in \mathcal{C}^1(\mathbb{R})$ for a.e. $x \in \Omega$, each function $u_{i,\beta}$ is a smooth solution to the Euler-Lagrange equation

$$(-\Delta)^{1/2} u_{i,\beta} + f_i(x,u_{i,\beta}) + e'(u_{i,\beta} - \bar{u}_i)u_{i,\beta} = \gamma_{i,\beta} u_{i,\beta} - \beta u_{i,\beta} \sum_{j\neq i} u_{j,\beta}^2$$

in Ω together with the boundary condition $u_i \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. (here $f_i(x,s) := \partial_s F_i(x,s)$). The Lagrange multipliers $\gamma_{i,\beta}$ are bounded uniformly with respect to β.

Proof. The first conclusion follows from the estimate

$$J^*_\beta(\mathbf{u}_\beta) \le J^*_\beta(\bar{\mathbf{u}}) = J(\bar{\mathbf{u}}) \le C$$

and the coercivity of J^*_β. Once the constraints are expressed through the Lagrange multipliers, the Euler-Lagrange equations can be derived classically, considering smooth variation of the minimizers $\mathbf{u}$. To conclude, testing the equation in $u_{i,\beta}$ by $u_{i,\beta}$ itself, the identity

$$\gamma_{i,\beta} = |u_{i,\beta}|^2_{H^{1/2}(\mathbb{R}^N)} + \int_\Omega u_{i,\beta}\left(f_{i,\beta}(x,u_{i,\beta}) + e'(u_{i,\beta}-\bar{u}_i)u_{i,\beta} + \beta u_{i,\beta}\sum_{j\neq i} u^2_{j,\beta}\right)\mathrm{d}x$$

yields the uniform bound for the multipliers. □

Corollary 2.7. *There exists a constant $C > 0$, independent of β, such that $\|\mathbf{u}_\beta\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^k)} \le C$.*

Proof. This a consequence of the Brezis-Kato inequality, suitably generalized to the fractional setting (see [2, Section 5], and in particular [2, Theorem 5.2]). We give a sketch of the proof of such result in the appendix. □

We are in a position to apply the result contained in [12], which implies a first uniform regularity estimate for the densities $\mathbf{u}_\beta$.

Theorem 2.8. *For any $\alpha < 1/2$, there exists a constant $C > 0$ which is independent of β, such that*

$$\|\mathbf{u}_\beta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N;\mathbb{R}^k)} \le C \qquad \forall \beta > 0.$$

In particular, the sequence $\mathbf{u}_\beta$ is compact in the $H^{1/2}_\Omega(\mathbb{R}^N;\mathbb{R}^k)$ topology and the uniform topology, and the limit $\bar{\mathbf{u}}$ of the family belongs to $\mathcal{C}^{0,\alpha}(\mathbb{R}^N;\mathbb{R}^k)$ for any $\alpha < 1/2$.

Proof. This is a direct consequence of [12, Theorem 1.3]. The only difference here is that the forcing term in the Euler-Lagrange equation (see Lemma 2.6) here depends also on the variable x. But the same proof of [12, Theorem 1.3] works also in this case, under the verified hypothesis there exists a constant $C > 0$ such that

$$\sup_{\beta>0} \|f_i(x,u_{i,\beta}) + e'(u_{i,\beta}-\bar{u}_i)u_{i,\beta} - \gamma_{i,\beta}u_{i,\beta}\|_{L^\infty(\Omega)} < C.$$ □

We can conclude with the optimal regularity result mentioned in the introduction.

Theorem 2.9 (Theorem 1.2)**.** *Under the previous assumptions, let us also suppose that*

(A) *each function F_i is independent of x, $F_i \in \mathcal{C}^{2,\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$ and $F_i'(0) = 0$.*

Then any minimizer $\mathbf{u}$ of J over the set $\mathbb{S}^k_{L^2}$ belongs to $\mathcal{C}^{0,1/2}(\mathbb{R}^N;\mathbb{R}^k)$ and satisfies the following Euler-Lagrange equation

$$u_i\left((-\Delta)^{1/2}u_i - F_i'(u_i)\right) = 0 \qquad \text{a.e. } in\ \Omega.$$

Proof. As of now, we have shown that the minimizer $\mathbf{u} \in C^{0,\alpha}(\mathbb{R}^N;\mathbb{R}^k)$ for any $\alpha < 1/2$ and that the approximating sequence $\mathbf{u}_\beta$ converges to $\mathbf{u}$ strongly in $H^{1/2}(\mathbb{R}^N;\mathbb{R}^k)$ and uniformly in $\mathbb{R}^N$. Passing to the limit in the Euler-Lagrange equation and using the uniform estimate in Lemma 2.6, we infer that $\mathbf{u}$ satisfies

$$\begin{cases} u_i u_j = 0 & \text{in } \Omega, \text{ for any } i \neq j \\ u_i\left((-\Delta)^{1/2}u_i - F_i'(u_i)\right) = 0 & \text{a.e. in } \Omega \\ u_i = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We are then in a position to apply the result in [12, Theorem 1.2] (see also [12, Proposition 9.2.]). □

3. Numerical simulations

We now present some numerical validations of the theoretical results obtained so far. In the following we shall present a numerical algorithm which is based on the approximation scheme developed in the previous section, which has then no pretensions of being the most suitable from a computational point of view. All the simulations were carried out with a finite element approximation scheme, using the free software FreeFem++, available at `http://www.freefem.org/ff++/`.

Let us consider a specific example, which has also a possible interest in the applied science. Let us consider the optimal partition in k subsets of the unit ball in $\mathbb{R}^2$, that is, the optimal partition induced by the functional

$$J(u_1, \ldots, u_k) := \begin{cases} \displaystyle\sum_{i=1}^{k} \frac{1}{2} |u_i|^2_{H^{1/2}(\mathbb{R}^2)} & \text{if } u_i \cdot u_j = 0 \text{ a.e. for every } j \neq i \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

constrained on the set $\mathbb{S}^k_{L^2}$ with $\Omega = B_1(0)$. Reasoning as in Section 2, we recall that the minimizers of the previous functional can be approximated by the minimizers of the approximating functional

$$J_\beta(u_1, \ldots, u_k) := \sum_{i=1}^{k} \frac{1}{2} |u_i|^2_{H^{1/2}(\mathbb{R}^2)} + \beta \int_{\mathbb{R}^N} \sum_{j<i} u_i^2 u_j^2$$

and, finally, the minimizers can be obtained as solutions to the Euler-Lagrange equation

$$(-\Delta)^{1/2} u_{i,\beta} + \beta u_{i,\beta} \sum_{j \neq i} u^2_{j,\beta} = \gamma_{i,\beta} u_{i,\beta} \tag{3.2}$$

for suitable Lagrange multipliers $\gamma_{i,\beta}$: a meta-algorithm inspired by this approximation is illustrated in Algorithm 1. Let us observe that, in order to find a solution to the nonlinear system of equations, we have used a fixed point method based on the steepest descent algorithm alternated with a projection on the constraint $\mathbb{S}^k_{L^2}$. Being the underlying problem strongly non-convex, no results about the convergence to the minimal solution are known, if not under the assumptions that the initial guess is already close to the optimal configuration. Similar results may be found for example in [1], where a different algorithm is presented to study the optimal partition problem of the standard Laplace-Dirichlet eigenvalues.

The only non trivial task in the algorithm is given by the non-local equation in u_i: to solve it, we can make use of the extensional formulation of the half-Laplacian (see [4] and reference therein), which relates the equation (3.2) to

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_+ \\ \partial_\nu v_i + \beta v_i \sum_{j \neq i} \bar{v}_j^2 = \gamma_i \bar{v}_i & \text{on } B_1 \times \{0\} \\ v_i \equiv 0 & \text{in } \mathbb{R}^2 \setminus B_1 \times \{0\} \end{cases} \tag{3.3}$$

where $v_i, \bar{v}_i \in H^1(\mathbb{R}^3_+)$ satisfy $v_i(\cdot, 0) = u_i$ and $\bar{v}_i(\cdot, 0) = \bar{u}_i$. The advantage of this formulations is that it can be readily approximated using finite element schemes, which are implement, for example, in the free software FreeFem++. To complete the approximating procedure, since (3.3) is defined on an unbounded set, we need

Algorithm 1 Approximating scheme

procedure ApproximatingProcedure
  initialize $\gamma_i, u_i, \bar{u}_i$
  $\beta \leftarrow 1$
  $\bar{\beta} \leftarrow$ a large constant
  repeat
    repeat
      Solve $(-\Delta)^{1/2} u_i + \beta u_i \sum_{j \neq i} \bar{u}_j^2 = \gamma_i \bar{u}_i,\ u_i \equiv 0$ in $\mathbb{R}^2 \setminus B_1$
      $\bar{u}_i \leftarrow \dfrac{\alpha u_i + (1-\alpha)\bar{u}_i}{\|\alpha u_i + (1-\alpha)\bar{u}_i\|_{L^2}}$ ▷ Projection on $\mathbb{S}^k_{L^2}$, $\alpha \in (0,1)$
      $\gamma_i \leftarrow |\bar{u}_i|^2_{H^{1/2}(\mathbb{R}^2)} + \beta \int_{\mathbb{R}^2} \bar{u}_i^2 \sum_{j \neq i} \bar{u}_j^2 \mathrm{d}x$
    until convergence in L^2 with a prescribed tolerance
    $\beta \leftarrow 2\beta$
  until $\beta > \bar{\beta}$ and convergence in L^2

to consider a bounding box $Q_L \subset \mathbb{R}^3_+$, $Q_L = (-L, L)^2 \times (0, 2L)$ with $L > 0$ large, and reformulated the equation as

$$\begin{cases} -\Delta v_i = 0 & \text{in } Q_L \\ \partial_\nu v_i + \beta v_i \sum_{j \neq i} \bar{v}_j^2 = \gamma_i \bar{v}_i & \text{on } (-L, L)^2 \times \{0\} \\ v_i \equiv 0 & \text{in } (-L, L)^2 \setminus B_1 \times \{0\} \end{cases}$$

for $v_i, \bar{v}_i \in H^1_{0,+}(Q_L) = \{w \in H^1_{0,+}(Q_L), w = 0 \text{ on } Q_L \setminus (-L, L)^2 \times \{0\}\}$. This last approximation is valid since, by the comparison, it is possible to show the solutions of equation (3.3) decay away from the origin $x = 0$. As a result, we can formulate the final Algorithm 2.

Algorithm 2 Approximating scheme revised

procedure ApproximatingProcedure
  $L, \bar{\beta} \leftarrow$ large constants
  initialize $v_i, \bar{v}_i \in H^1_{0,+}(Q_L)$, $\gamma_i \in \mathbb{R}$
  $\beta \leftarrow 1$
  repeat
    repeat
      Solve $\begin{cases} -\Delta v_i = 0 & \text{in } Q_L \\ \partial_\nu v_i + \beta v_i \sum_{j \neq i} \bar{v}_j^2 = \gamma_i \bar{v}_i & \text{on } (-L, L)^2 \times \{0\} \\ v_i \equiv 0 & \text{in } (-L, L)^2 \setminus B_1 \times \{0\} \end{cases}$
      $\bar{v}_i \leftarrow \dfrac{\alpha v_i + (1-\alpha)\bar{v}_i}{\|\alpha v_i + (1-\alpha)\bar{v}_i\|_{L^2((-L,L)^2 \times \{0\})}}$
      $\gamma_i \leftarrow |\bar{v}_i|^2_{H^1_{0,+}(Q_L)} + \beta \int_{(-L,L)^2 \times \{0\}} \bar{v}_i^2 \sum_{j \neq i} \bar{v}_j^2 \mathrm{d}x$
    until convergence in L^2 of the traces up to a prescribed tolerance
    $\beta \leftarrow 2\beta$
  until $\beta > \bar{\beta}$ and convergence in L^2 of the traces

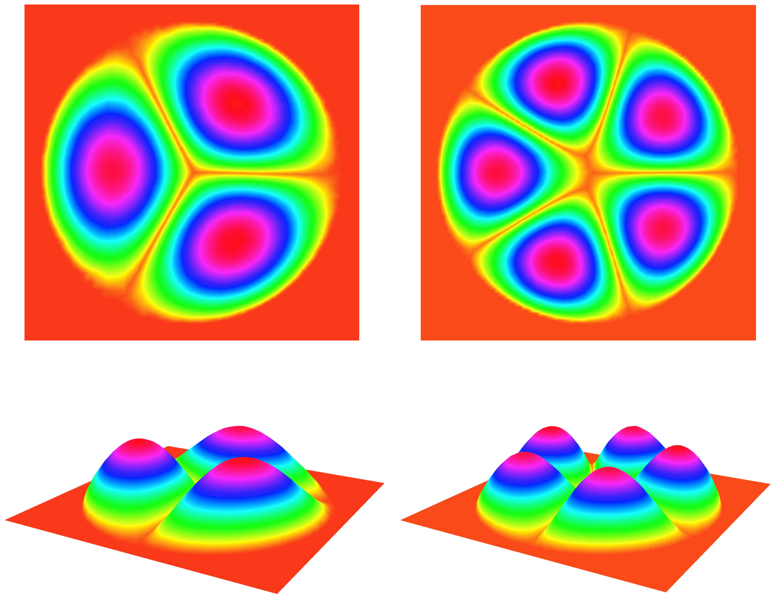

FIGURE 1. Optimal partitions related to the problem 3.1. The non complete symmetry of the solutions, which can be expected by comparison with the partition problem involving the standard Laplacian, may be an effect of the presence of the bounding box Q_L: the more L is chosen large, the more such effect should be smoothed out. In any case, even for large values of L this phenomenon seems persistent.

Remark 3.1. It should be mentioned that, though the extensional formulation of the fractional Laplacian alleviates us from solving non-local equations, it transforms N-dimensional optimal partition problems in to $N+1$-dimensional boundary partition problems. For example, a planar problem is solved resorting to a fully three-dimensional one. Since both three-dimensional partition problem and, in general, boundary problems are stiff from a numerical point of view, it may seem surprising that the algorithms presented in this section converge in general, with just simple tunings of the parameters.

Remark 3.2. In order to obtain more accurate solutions, but sacrificing the efficiency, we have also inserted a step involving mesh-refinements.

As a result of the numerical simulation, we collect in Figure 1 the solutions obtained for the problem (3.1) in the case of $k = 3$ components and $k = 5$ components. In Figure 2 we show the corresponding solutions in the case of the standard Laplace operator: comparing the two situations, it is possible to see that, even though qualitatively very similar, the solutions *may* be different not only with respect to the regularity of their respective densities, but also in the geometry of the sets constituting the partition.

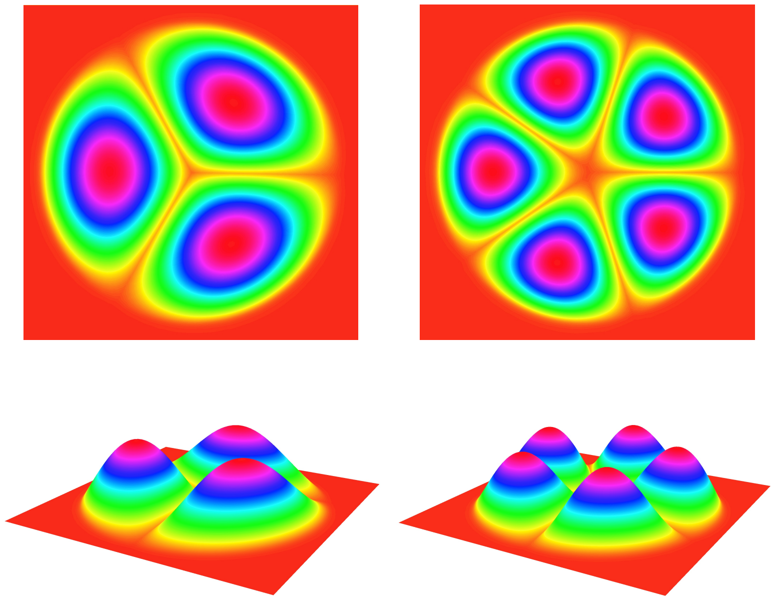

FIGURE 2. Optimal partitions related to the problem 3.1, in the case the standard Laplace operator, obtained with the same approximating scheme employed for the half-Laplacian (see also [6] for further examples). The solutions are qualitatively similar, even though in this former case the transition between two different densities is smoother (in particular, solution are Lipschitz-continuous, as shown in [10])

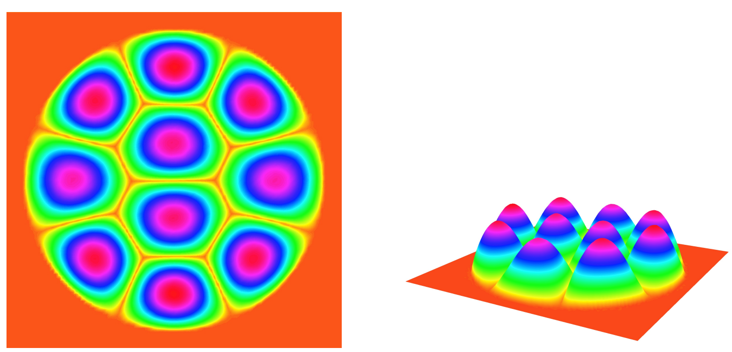

FIGURE 3. Optimal partitions related to the problem 3.1, in the case of $k = 10$ components. It is tempting to extend the *hexagonal conjecture* (see for instance [5, 1]) also to the non-local setting.

APPENDIX A. THE BREZIS-KATO INEQUALITY

In this last section, we will give a proof of Corollary 2.7, using in fact the following version of the Brezis-Kato inequality

Lemma A.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain and let us consider $\mathbf{u} \in H^{1/2}_\Omega(\mathbb{R}^N, \mathbb{R}^k)$ to be solutions to the system*

$$(-\Delta)^{1/2} u_i = a_i(1 + |u_i|) - \beta u_i \sum_{j \neq i} u_j^2. \tag{A.1}$$

where $a_i \in L^N(\mathbb{R}^N)$. Then $u_i \in L^\infty(\mathbb{R}^N)$ for all $i = 1, \dots, k$ and the norm can be bounded uniformly in β with a constant that depends only on the $H^{1/2}$-norm of $\mathbf{u}$ and the L^N-norm of a_i.

Remark A.2. In order to apply the previous result to the setting of Corollary 2.7, it is sufficient to introduce the functions

$$a_{i,\beta} := \frac{(\gamma_{i,\beta} - Ke'(u_{i,\beta} - \bar{u}_i))u_{i,\beta} - f_i(x, u_{i,\beta})}{1 + |u_{i,\beta}|}$$

and to observe that, thanks to the sub-criticality of f_i and the uniform boundedness of Lagrange multipliers, we have $\|a_{i,\beta}\|_{L^N(\mathbb{R}^N)} \leq C$ uniformly in β.

Proof. In order to simplify the proof, we resort to the extensional formulation of the half-Laplacian, relating the system (A.1) to

$$\begin{cases} -\Delta v_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu v_i = a_i(1 + |v_i|) - \beta v_i \sum_{j \neq i} v_j^2 & \text{on } \Omega \subset \partial\mathbb{R}^{N+1}_+ \\ v_i = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $v_i \in H^1(\mathbb{R}^{N+1}_+)$ satisfy $v_i(\cdot, 0) = u_i$. Let $g_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ be a smooth approximation of the modulus functions, that is, $g_\varepsilon(t) = \sqrt{\varepsilon + t^2}$. The Stampacchia's lemma and the Lebesgue's theorem ensure that

$$g_\varepsilon(v_i) \to |v_i| \text{ in } H^1(\mathbb{R}^{N+1}_+), \quad g'_\varepsilon(v_i)v_i \to |v_i| \text{ in } L^2(\mathbb{R}^N)$$

For any test function $\phi \in H^1(\mathbb{R}^{N+1}_+)$ such that $\phi \geq 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+} \nabla g_\varepsilon(v_i) \nabla \phi + \int_{\mathbb{R}^N} \beta g'_\varepsilon(v_i) v_i \sum_{j \neq i} v_j^2 \phi \\ = \int_{\mathbb{R}^N} g'_\varepsilon(v_i) a_i (1 + |v_i|)\phi - \int_{\mathbb{R}^{N+1}_+} g''_\varepsilon(v_i) |\nabla v_i|^2 \phi \end{aligned}$$

and letting $\varepsilon \to 0^+$ we obtain

$$\int_{\mathbb{R}^{N+1}_+} \nabla |v_i| \nabla \phi + \int_{\mathbb{R}^N} \beta |v_i| \sum_{j \neq i} v_j^2 \phi \leq \int_{\mathbb{R}^N} \operatorname{sgn}(v_i) a_i (1 + |v_i|)\phi.$$

(similar computations are present in [12, Lemma 5.5]). As a result, each $|v_i| \in H^1(\mathbb{R}^{N+1}_+)$ is a subsolution of the equation in $w_i \in H^1(\mathbb{R}^{N+1}_+)$

$$\begin{cases} -\Delta w_i = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ \partial_\nu w_i = |a_i|(1 + w_i) & \text{on } \Omega \subset \partial\mathbb{R}^{N+1}_+ \\ w_i = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

Thus, if we show a uniform bound for the functions w_i in L^∞, by the comparison principle we could evince that the same bounds holds for the functions $|v_i|$. To conclude it is then sufficient to recall the Brezis-Kato estimate for the half-Laplacian, shown in [2, Theorem 5.2], which implies the sought L^∞ bound. $\square$

Acknowledgement

The author is indebted with the anonymous referee for suggesting many useful improvements to the original manuscript.

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