MULTIDIMENSIONAL ENTIRE SOLUTIONS FOR AN ELLIPTIC SYSTEM MODELLING PHASE SEPARATION

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Abstract. For the system of semilinear elliptic equations

$$\Delta V_i = V_i \sum_{j \neq i} V_j^2, \qquad V_i > 0 \qquad \text{in } \mathbb{R}^N$$

we devise a new method to construct entire solutions. The method extends the existence results already available in the literature, which are concerned with the 2-dimensional case, also in higher dimensions $N \geq 3$. In particular, we provide an explicit relation between orthogonal symmetry subgroups, optimal partition problems of the sphere, the existence of solutions and their asymptotic growth. This is achieved by means of new asymptotic estimates for competing system and new sharp versions for monotonicity formulae of Alt-Caffarelli-Friedman type.

1. Introduction

The elliptic systems

$$\begin{cases} \Delta V_i = V_i \sum_{j \neq i} V_j^2 \\ V_i \geq 0 \end{cases} \qquad \text{in } \mathbb{R}^N,\ i = 1, \dots, k, \tag{1.1}$$

which arise in the blow-up analysis of phase-separation phenomena in coupled Schrödinger equations [1, 2, 11], has attracted an increasing attention in the last years, and by now many results concerning existence and qualitative properties of the solutions are available. In this paper we go further in the analysis, proving the existence of N-*dimensional solutions* to (1.1) in $\mathbb{R}^N$ for any $N \geq 2$. With this, we mean that we construct solutions in $\mathbb{R}^N$ which cannot be obtained from solutions in lower dimension by adding the dependence on some "mute" variable. Our results extend the construction developed in [2], which concerns the planar case $N = 2$. In this perspective, we mention that previous results contained in [1, 2] only regard the existence of solutions in dimension $N = 1$ or 2, and the question of the existence in higher dimension is open.

In order to state our main results, we introduce some notation. We denote by $\mathcal{O}(N)$ the orthogonal group of $\mathbb{R}^N$, and by $\mathfrak{S}_k$ the symmetric group of permutations of $\{1, \dots, k\}$. Let us assume that there exists a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$, where $\mathcal{G} < \mathcal{O}(N)$ is a nontrivial subgroup. We define the *equivariant action* of $\mathcal{G}$ on

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$H^1(\mathbb{R}^N, \mathbb{R}^k)$ in the following way:

$$\begin{aligned} \mathcal{G} \times H^1(\mathbb{R}^N, \mathbb{R}^k) &\to H^1(\mathbb{R}^N, \mathbb{R}^k) \\ (g, \mathbf{u}) &\mapsto g \cdot \mathbf{u} := \left(u_{(h(g))^{-1}(1)} \circ g, \dots, u_{(h(g))^{-1}(k)} \circ g\right) \end{aligned} \tag{1.2}$$

where $\circ$ denotes the usual composition of functions, and we used the vector notation $\mathbf{u} := (u_1, \dots, u_k)$. The set

$$H_{(\mathcal{G},h)} := \left\{\mathbf{u} \in H^1(\mathbb{R}^N, \mathbb{R}^k) : \mathbf{u} = g \cdot \mathbf{u} \ \forall g \in \mathcal{G}\right\}$$

is the subspace of the $(\mathcal{G}, h)$-equivariant functions.

Definition 1.1. For $k \in \mathbb{N}$, a nontrivial subgroup $\mathcal{G} < \mathcal{O}(N)$, and a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$, we write that the triplet $(k, \mathcal{G}, h)$ is admissible if there exists a $(\mathcal{G}, h)$-equivariant function $\mathbf{u}$ with the following properties:

- *(i)* $u_i \geq 0$ and $u_i \not\equiv 0$ for every i;
- *(ii)* $u_i u_j \equiv 0$ for every $i \neq j$;
- *(iii)* for $i = 2, \dots, k$ there exists $g \in \mathcal{G}$ such that
$$u_i = u_1 \circ g.$$

Remark 1.2. Notice that, if $(k, \mathcal{G}, h)$ is admissible triplet, then all the $(\mathcal{G}, h)$-equivariant functions satisfy *(iii)* with the same i and g. In particular, they are completely determined by their first component. We also underline the fact that there may exist symmetries in $\mathcal{G}$ whose corresponding permutation is the identity. In this case, these symmetries are then imposed on the functions themselves.

Let $(k, \mathcal{G}, h)$ be an admissible triplet. We denote by

$$\Lambda_{(\mathcal{G},h)} := \left\{ \varphi \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k) \;\middle|\; \begin{array}{l} \varphi \text{ is the restriction on } \mathbb{S}^{N-1} \text{ of a} \\ (\mathcal{G}, h)\text{-equivariant function fulfilling} \\ (i)\text{-}(iii) \text{ in Definition 1.1} \end{array} \right\}. \tag{1.3}$$

We consider the minimization problem

$$\ell_{(k,\mathcal{G},h)} := \inf_{\varphi \in \Lambda_{(\mathcal{G},h)}} \frac{1}{k} \sum_{i=1}^k \left(\sqrt{\left(\frac{N-2}{2}\right)^2 + \frac{\int_{\mathbb{S}^{n-1}} |\nabla_\theta \varphi_i|^2}{\int_{\mathbb{S}^{n-1}} \varphi_i^2}} - \frac{N-2}{2} \right), \tag{1.4}$$

where ∇_θ denotes the tangential gradient on $\mathbb{S}^{N-1}$.

Theorem 1.3. *For any admissible pair $(k, \mathcal{G}, h)$, there exists a solution $\mathbf{V}$ of (1.1) with k components in $\mathbb{R}^N$ satisfying the following properties:*

- $\mathbf{V}$ *is $(\mathcal{G}, h)$-equivariant;*
- *it results*

$$\lim_{r \to +\infty} \frac{1}{r^{N-1+2\ell_{(k,\mathcal{G},h)}}} \int_{\partial B_r} \sum_{i=1}^k V_i^2 \in (0, +\infty). \tag{1.5}$$

Since the theorem is quite general, we think that it is worth to spend some time making some explicit examples. This will be done in the last part of the introduction. For the moment, we anticipate that with our result we can both recover Theorem 1.3 and 1.6 in [2], and moreover we can produce a wealth of new solutions existing only in dimension $N \geq 3$.

We also observe that condition (1.5) establishes that the solution $\mathbf{V}$ grows at infinity, in quadratic mean, like the power $|x|^{\ell_{(k,\mathcal{G},h)}}$. It is worth to remark that for any solution to (1.1) *having algebraic growth*, that is satisfying the point-wise upper bound

$$V_1(x) + \cdots + V_k(x) \le C(1 + |x|^d) \qquad \forall x \in \mathbb{R}^N$$

for some $C, d \ge 1$, it is possible to defined the *growth rate* as the uniquely determined value $d \in (0, +\infty)$ such that

$$\lim_{r \to +\infty} \frac{1}{r^{N-1+m}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2 = \begin{cases} +\infty & \text{if } m < d \\ 0 & \text{if } m > d, \end{cases}$$

see Proposition 1.5 in [8]. Theorem 1.3 gives then a sharp version of the previous inequality. Not only it specifies the growth rate of the function ($d = \ell(k, \mathcal{G}, h)$) but it also states that for this precise growth rate, the previous limit is positive and finite. In this perspective we can prove that the solutions of Theorem 1.3 have minimal growth rate among all the possible $(\mathcal{G}, h)$-equivariant solutions.

Theorem 1.4. *Let $(k, \mathcal{G}, h)$ be an admissible pair, and let $\mathbf{V}$ be a $(\mathcal{G}, h)$-equivariant solution of (1.1). Then the growth rate of $\mathbf{V}$ is at least $\ell(k, \mathcal{G}, h)$.*

Both the proofs of Theorems 1.3 and 1.4 exploit the hidden relationship between the elliptic system (1.1) and optimal partition problems of type (1.4). This relationship arises for instance by means of the validity of the following Alt-Caffarelli-Friedman monotonicity formula for $(\mathcal{G}, h)$-equivariant solutions.

For $\mathbf{V} \in H^1(\mathbb{R}^N, \mathbb{R}^k)$ and $i = 1, \dots, k$ we define

$$J_i(r) := \int_{B_r} \frac{|\nabla V_i|^2 + V_i^2 \sum_{j \ne i} V_j^2}{|x|^{N-2}}.$$

Proposition 1.5. *Let $(k, \mathcal{G}, h)$ be an admissible triplet. There exists a constant $C > 0$ depending only on N and on $(k, \mathcal{G}, h)$ such that, for any $(\mathcal{G}, h)$-equivariant solution $\mathbf{V}$ of (1.1), the function*

$$r \mapsto \frac{1}{r^{2k\ell(k,\mathcal{G},h)}} e^{-Cr^{-1/2}} J_1(r) \cdots J_k(r)$$

is monotone non-decreasing for $r > 1$; we recall that $\ell(k, \mathcal{G}, h)$ has been defined in (1.4).

We review now the main known results regarding entire solutions to system (1.1) which were already available, starting with the $k = 2$ components system. The 1-dimensional problem was studied in [1], where it is proved that there exists a solution satisfying the symmetry property $V_2(x) = V_1(-x)$, the monotonicity condition $V_1' > 0$ and $V_2' < 0$ in $\mathbb{R}$, and having at most linear growth, in the sense that there exists $C > 0$ such that

$$V_1(x) + V_2(x) \le C(1 + |x|) \qquad \forall x \in \mathbb{R}^N.$$

Up to translations, scaling, and exchange of the components, this is the unique solution in dimension $N = 1$, see [2, Theorem 1.1]. The linear growth is the minimal admissible growth for non-constant positive solutions of (1.1). Indeed, in any dimension $N \ge 1$, if (V_1, V_2) is a *nonnegative* solution of (1.1) (which means that the condition $V_i > 0$ is replaced by $V_i \ge 0$) and satisfies the sublinear growth condition

$$V_1(x) + V_2(x) \le C(1 + |x|^\alpha) \qquad \text{in } \mathbb{R}^N$$

for some $\alpha \in (0,1)$ and $C > 0$, then one between V_1 and V_2 is 0, and the other has to be constant. This *Liouville-type theorem* has been proved by B. Noris et al. in [7, Propositions 2.6].

Differently from the problem in $\mathbb{R}$, in dimension $N = 2$, and hence in any dimension $N \geq 2$, system (1.1) with $k = 2$ has infinitely many "geometrically distinct" solutions, i.e. solutions which cannot be obtained one from the other by means of rigid motions, scalings, or exchange of the components, see [2, Theorem 1.3] and [10, Theorems 1.1 and 1.5]. These solutions can be distinguished according to their growth rates and symmetry properties. In particular, in [2] the authors proved the existence of solutions having algebraic growth, while the results in [10] concern solutions having exponential growth in x and being periodic in y.

Regarding systems with several components, the aforementioned existence results admit analogue counterparts for any $k \geq 3$, see [2, Theorem 1.6] and [10, Theorem 1.8].

It is important to stress that the proofs in [2, 10] use the fact that the problem is posed in dimension $N = 2$, and apparently cannot be extended to higher dimension (see the forthcoming Remark 4.4 for a more detailed discussion).

In parallel to the existence results, great efforts have been devoted to the analysis of the 1-dimensional symmetry of solutions under suitable assumptions; this, as explained in [1], is inspired by some analogy in the derivation of (1.1) and of the Allen-Chan equation, for which symmetry results in the spirit of the celebrated De Giorgi's conjecture have been widely studied. In this context, we recall that assuming $k = 2$ and $N = 2$, A. Farina proved that if (V_1, V_2) has algebraic growth and $\partial_2 V_1 > 0$ in $\mathbb{R}^2$, then (V_1, V_2) is 1-dimensional [5]. In the higher dimensional case $N \geq 2$ with $k = 2$, A. Farina and the first author proved a Gibbons-type conjecture for system (1.1), see [6]. Furthermore, as product of the main results in [13, 14], K. Wang showed that any solution of (1.1) with $k = 2$ has linear growth is 1-dimensional. We mention also [1, Theorem 1.8] and [2, Theorem 1.12], which are now included in the Wang's result.

As far as the 1-dimensional symmetry for systems with $k > 2$ is concerned, we refer to [8, Theorem 1.3], where the main results in [6, 13, 14] are extended to systems with many components by means of improved Liouville-type theorems for multi-components systems, which put in relation the number of nontrivial components for a nonnegative solution of the first equation in (1.1) and its growth rate. In this perspective, Theorem 1.4 is the counterpart of [8, Theorem 1.7] in a $(\mathcal{G}, h)$-equivariant setting. As a product of these two results, we can also derive the following corollary.

Corollary 1.6. *For $k, N \in \mathbb{N}$, let*

$$\mathcal{L}_k(\mathbb{S}^{N-1}) := \inf_{(\omega_1,\dots,\omega_k)\in\mathcal{P}_k} \sup_{i=1,\dots,k} \lambda_1(\omega_i),$$

where $\mathcal{P}_k$ is the set of partitions of $\mathbb{S}^{N-1}$ in k open disjoint sets, and λ_1 denotes the first eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Let also $(k, \mathcal{G}, h)$ be any admissible triplet, with $\mathcal{G} < \mathcal{O}(N)$. Then

$$\mathcal{L}(\mathbb{S}^{N-1}) \leq \ell(k, \mathcal{G}, h).$$

It is tempting to conjecture that equality holds for an appropriate choice of $(\mathcal{G}, h)$, at least for some values of k, N. Indeed, in light of the known results in the literature, this is the case for $k = 2$ and $k = 3$, for every N.

To conclude, we mention also the contribution [15], where the authors considered the fractional analogue of (1.1). Such problem exhibit new interesting phenomena with respect to the local case. Moreover, we observe that our results, as those in [2], seem to be somehow connected with those in [16], which on the other hand concern finite energy decaying solutions of a different problem.

1.1. **A wealth of new solutions: applications of Theorem 1.3.** We recalled that, for any $k \geq 2$, problem (1.1) has several solutions in $\mathbb{R}^2$. Clearly, these are also solutions in higher dimension, and up to now it was an open question whether or not there exist solutions of (1.1) in $\mathbb{R}^N$ with $N \geq 3$ which cannot be obtained by solutions in $\mathbb{R}^2$. Theorem 1.3 gives a positive answer. In what follows we show several applications of Theorem 1.3, showing that this result can be used as a recipe to construct entire solutions of (1.1).

A concrete example in $\mathbb{R}^3$ for $k = 2$. To start with a very concrete example, we focus on problem (1.1) in $\mathbb{R}^3$ with $k = 2$, and we examine the case where $\mathcal{G}$ is equal to the group of symmetries generated by the reflections T_1, T_2, T_3 with respect to the planes $\{x = 0\}$, $\{y = 0\}$, and $\{z = 0\}$ respectively, and $h : \mathcal{G} \to \mathfrak{S}_k$ is defined on the generators of $\mathcal{G}$ by $h(T_i) = (1\ 2)$ for every i. We used here the standard notation $(1\ 2)$ to denote the cycle mapping 1 in 2, and 2 in 1. In order to check that this is an admissible triplet, we verify that

$$(u_1, u_2) = \big((xyz)^+, (xyz)^-\big)$$

is a $(\mathcal{G}, h)$-equivariant function satisfying (i)-(iii) in Definition 1.1. Indeed

$$T_i \cdot (u_1, u_2) = (\text{see (1.2)}) = (u_2 \circ T_i, u_1 \circ T_i) = (\text{def. } \mathbf{u}) = (u_1, u_2),$$

for every i, and since $\mathcal{G}$ is generated by T_1, T_2, T_3, this is sufficient to conclude that $\mathbf{u}$ is $(\mathcal{G}, h)$-equivariant. As a consequence, by Theorem 1.3 there exists a $(\mathcal{G}, h)$-equivariant solution of (1.1) in $\mathbb{R}^3$ with $k = 2$. It is not difficult to check that, since the symmetries of $\mathcal{G}$ involve the 3 variables, this solution cannot be obtained by a 2-dimensional solution adding the dependence of 1-variable. Indeed, by Theorem 1.4 in [2] the blow-down sequence

$$(u_R(x), v_R(x)) := \left(\frac{1}{R^{N-1}} \int_{\partial B_R} u^2 + v^2\right)^{-1/2} (u(Rx), v(Rx))$$

converges, in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, as $R \to +\infty$ and up to a subsequence, to a pair (Ψ^+, Ψ^-), where Ψ is a homogeneous harmonic polynomial of integer degree $\ell(2, \mathcal{G}, h)$. By minimality

$$\ell(2, \mathcal{G}, h) \leq \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^+|^2}{\int_{\mathbb{S}^2} |(xyz)^+|^2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^-|^2}{\int_{\mathbb{S}^2} |(xyz)^-|^2}} - \frac{1}{2}\right),$$

and the right hand side is equal to 3: indeed, since $\Phi := xyz$ is a homogeneous harmonic polynomial of degree 3, its radial part $\Phi|_{\mathbb{S}^2}$ solves

$$-\Delta_\theta \Phi|_{\mathbb{S}^2} = 12 \Phi|_{\mathbb{S}^2} \qquad \text{in } \mathbb{S}^2,$$

and this permits to carry on explicit computations. This means that Ψ is a homogeneous harmonic polynomial of degree $\ell(2, \mathcal{G}, h) \leq 3$, and by $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ convergence, is $(\mathcal{G}, h)$-equivariant. It is then necessary that $\Psi = \Phi = xyz$, and, as Ψ is the uniform limit of scaling of the function $u - v$, this ensures that (u, v) is 3-dimensional, and moreover we have also that it has growth rate equal to 3.

General case in $\mathbb{R}^N$ with $k = 2$. The very same argument as before can be considered by taking any homogeneous harmonic polynomial Φ in $\mathbb{R}^N$ of degree $d \in \mathbb{N}$, with a nontrivial finite group of symmetry $\mathcal{G}$. Let $T_1, \dots, T_m$ be a system of generators of $\mathcal{G}$. To any T_i we associate the cycle $(1\ 2)$. This induces a homomorphism $h : \mathcal{G} \to \mathfrak{S}_2$, and it is not difficult to check that $(2, \mathcal{G}, h)$ is an admissible triplet. Theorem 1.3 yields the existence of a $(\mathcal{G}, h)$-equivariant solution (u, v) to (1.1) with growth rate $\ell(2, \mathcal{G}, h) \le d$, with equality whenever Φ is the homogeneous harmonic polynomial having minimal growth rate among all the $(\mathcal{G}, h)$-equivariant homogeneous harmonic polynomials. By choosing Φ which fulfils this requirement from the beginning, the previous discussion permits to deduce that (u, v) is N-dimensional.

In particular, we emphasize that for the case $N = 2$, where homogeneous harmonic polynomials are of type $\mathfrak{Re}((x + iy)^d)$, we recover [2, Theorem 1.3].

The case $k \ge 3$ in $\mathbb{R}^2$. Let $k \ge 3$, and for any $h \in \mathbb{N}$ let $d = hk/2$. We denote by R_d the rotation of angle π/d, by T_y the reflection with respect to $\{y = 0\}$ (this corresponds to consider complex conjugation in $\mathbb{C}$), and we consider the group $\mathcal{G} < \mathcal{O}(N)$ generated by R_d and T_y. We define a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$ (the group of permutations of $\{1, \dots, k\}$) letting

$$h(R_d) := (1\ 2\ \cdots\ d) \quad \text{and} \quad h(T_y) : i \mapsto k + 1 - i,$$

where the indexes are counted modulus k. Then $(k, \mathcal{G}, h)$ is an admissible triplet. Indeed, the function

$$\begin{aligned} u_1 &:= \begin{cases} r^d \cos\theta & \text{in } \bigcup_{i=0}^{h-1} R_d^{ik}(\{-\pi/2d < \theta < \pi/2d\}) \\ 0 & \text{otherwise} \end{cases} \\ u_2 &:= u_1 \circ R_d \\ &\ \vdots \\ u_k &:= u_{k-1} \circ R_d = u_1 \circ R_d^{k-1} \end{aligned}$$

is $(\mathcal{G}, h)$-equivariant and satisfies (i)-(iii) in Definition 1.1, and the corresponding solution given by Theorem 1.3 was already found by [2].

The general case $k \ge 3$ in $\mathbb{R}^3$. The case $k \ge 3$ and $N \ge 3$ is intrinsically more involved, and hence we focus on some particular examples given by the group of symmetry of the Platonic polyhedra. Let us consider for instance the group $\mathcal{G} < \mathcal{O}(N)$ associated to the tetrahedron $\mathcal{T}$. It is known that this group is isomorphic to $\mathfrak{S}_4$ through an isomorphism h. In order to define the function φ satisfying (i)-(iii) of Definition 1.1, we first take a tetrahedron with barycenter in 0, and define on a face A a positive function $\tilde\varphi_1$ being 0 on ∂A, and being symmetric with respect to all the transformations in $\mathcal{G}$ leaving invariant A. By rotation, we can define $\tilde\varphi_2$, $\tilde\varphi_3$ and $\tilde\varphi_4$ on the remaining faces. Now, considering the radial projection of the tetrahedron into the unit sphere $\mathbb{S}^2$, we obtain a function $(\varphi_1, \dots, \varphi_4)$ whose 1-homogeneous extension is $(\mathcal{G}, h)$-equivariant and satisfies (i)-(iii) of Definition 1.1. Thus $(4, \mathcal{G}, h)$ is an admissible triplet, and Theorem 1.3 yields the existence of a $(\mathcal{G}, h)$-equivariant solution for the system with 4 components in $\mathbb{R}^3$. Since the symmetries of the tetrahedron involve the dependence on 3 variables, this solution is not 2-dimensional. In a similar way, one can construct equivariant solutions with respect to the symmetries of the cube for systems with $k = 3$ or $k = 6$ components, equivariant solutions with respect to the symmetries of the octahedron for systems with $k = 2$ (this is actually the first example we considered), $k = 4$ or $k = 8$ components, and so on.

2. Preliminaries

We introduce some notation and review some known results. Let $\beta > 0$, and let $\mathbf{U}$ be a solution to

$$\begin{cases} \Delta U_i = \beta U_i \sum_{j\neq i} U_j^2 & \text{in } B_R \\ U_i > 0 & \text{in } B_R. \end{cases} \tag{2.1}$$

For $0 < r < R$, we set

- $H(\mathbf{U}, r) := \dfrac{1}{r^{N-1}} \displaystyle\int_{\partial B_r} \sum_{i=1}^{k} U_i^2$
- $E(\mathbf{U}, r) := \dfrac{1}{r^{N-2}} \displaystyle\int_{B_r} \sum_{i=1}^{k} |\nabla U_i|^2 + \beta \sum_{1\leq i<j\leq k} U_i^2 U_j^2$
- $N(\mathbf{U}, r) := \dfrac{E(\mathbf{U}, r)}{H(\mathbf{U}, r)}$ Almgren frequency function.

Under the previous notation, by Proposition 5.2 in [2] it is known that $N(\mathbf{U}, \cdot)$ is monotone non-decreasing for $0 < r < R$,

$$\frac{d}{dr} H(\mathbf{U}, r) = \frac{2}{r} E(\mathbf{U}, r) + \frac{2\beta}{r^{N-1}} \int_{B_r} \sum_{i<j} U_i^2 U_j^2 > 0,$$

and for any such r

$$\int_1^r 2\beta \frac{\int_{B_s} \sum_{i<j} U_i^2 U_j^2}{s^{N-1} H(\mathbf{U}, s)}\, ds \leq N(r). \tag{2.2}$$

The frequency function gives information about the behaviour of the solutions with respect to radial dilations. In this perspective, the previous monotonicity formula tells us that solutions of (1.1) are radially increasing. With some extra work, one can prove much more: for any solution having algebraic growth there exists a value $d \in (0, +\infty)$ such that

$$\lim_{r\to+\infty} \frac{\frac{1}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2}{r^{2d'}} = \begin{cases} +\infty & \text{if } d' < d \\ 0 & \text{if } d' > d; \end{cases} \tag{2.3}$$

moreover, d can be characterized as

$$d = \lim_{r\to+\infty} N(\mathbf{V}, r),$$

see [8, Proposition 1.5]. Notice that on the left hand side of (2.3) we have the quadratic average of $\mathbf{V}$ on spheres of increasing radius divided by a power of r^2. Therefore, the previous proposition tells us that any solution of (1.1) grows, radially, almost like the power r^d. For this reason we write that $\mathbf{V}$ *has growth rate* d if (2.3) holds.

In the previous discussion $\beta > 0$ was fixed. Let us now consider a sequence of parameters $\beta \to +\infty$, and a corresponding sequence $\{\mathbf{U}_\beta\}$ of solutions to (2.1). The asymptotic behaviour of the family $\{\mathbf{U}_\beta\}$ has been studied in a number of papers [1, 4, 7, 9, 11, 12, 17], and many results are available. We only recall that, if the sequence is bounded in $L^\infty(B_R)$, then it is in turn uniformly bounded in

$\mathrm{Lip}(B_R)$, and hence up to a subsequence it converges to a limit $\mathbf{U}$ in $\mathcal{C}^{0,\alpha}(B_R)$ and in $H^1_{\mathrm{loc}}(B_R)$ (see [9, 7]). If $\mathbf{U} \not\equiv \mathbf{0}$, then $\mathbf{U}$ is Lipschitz continuous and $\{\mathbf{U} = \mathbf{0}\}$ has Hausdorff dimension $N-1$. Moreover, $H(\mathbf{U}, r)$ is non-decreasing and is $\neq 0$ for every $r > 0$ (see [12]).

3. An Alt-Caffarelli-Friedman monotonicity formula for equivariant solutions

In the rest of the section we aim at proving Proposition 1.5. We always suppose that $(k, \mathcal{G}, h)$ is an admissible triplet, according to Definition 1.1. Moreover, we often omit the mention "up to a subsequence" for simplicity. The proof is divided in several steps, and, as usual when dealing with Alt-Caffarelli-Friedman monotonicity formulae for competing systems, is based upon a control on an "approximated" optimal partition problem on $\mathbb{S}^{N-1}$. For any $\mathbf{u} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, we let

$$I_\beta(\mathbf{u}) := \frac{1}{k} \sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_i|^2 + \frac{1}{2}\beta u_i^2 \sum_{j \neq i} u_j^2}{\int_{\mathbb{S}^{n-1}} u_i^2} \right),$$

where

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right).$$

We denote by $\hat{H}_{(\mathcal{G},h)}$ the subspace of $(\mathcal{G},h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, and we introduce the optimal value

$$\ell_\beta(k, \mathcal{G}, h) := \inf_{\hat{H}_{(\mathcal{G},h)}} I_\beta.$$

In what follows, to keep the notation as simple as possible, we simply write ℓ and ℓ_β instead of $\ell(k, \mathcal{G}, h)$ and $\ell_\beta(k, \mathcal{G}, h)$, respectively.

Lemma 3.1. *Both ℓ and ℓ_β are positive and achieved (for all $\beta > 0$). It results $\ell_\beta \to \ell$ as $\beta \to +\infty$, and there exists a minimizer for ℓ_β which solves*

$$\begin{cases} -\Delta_\theta u_{i,\beta} = \lambda_\beta u_{i,\beta} - \beta u_{i,\beta} \sum_{j \neq i} u_j^2 & \text{in } \mathbb{S}^{N-1} \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1} \\ \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 = 1 & \forall i, \end{cases} \tag{3.1}$$

where $\lambda_\beta \in \mathbb{R}$, and Δ_θ denotes the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Moreover, $\mathbf{u}_\beta \rightharpoonup \varphi$ weakly in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, and φ is a nonnegative minimizer for ℓ.

Proof. Restricting ourselves to the subset of functions in $\hat{H}_{(\mathcal{G},h)}$ whose components have prescribed $L^2(\mathbb{S}^{N-1})$-norm equal to 1, it is easy to check that the functional I_β is weakly lower semi-continuous and coercive. Since $\hat{H}_{(\mathcal{G},h)}$ is also weakly closed, the direct method of the calculus of variations ensures the existence of a minimizer $\mathbf{u}_\beta$ for ℓ_β, which can be assumed to be nonnegative. By the Palais' principle of symmetric criticality (notice that I_β is invariant under the action of any symmetry in $\mathcal{O}(N)$), the Lagrange multipliers rule, and the strong maximum principle, it follows that $\mathbf{u}_\beta$ satisfies

$$\begin{cases} -\Delta_\theta u_{i,\beta} + \sum_{j \neq i} \frac{1}{2}\left(1 + \frac{\mu_{j,\beta}}{\mu_{i,\beta}}\right) \beta u_{i,\beta} u_{j,\beta}^2 = \lambda_{i,\beta} u_{i,\beta} & \text{in } \mathbb{S}^{N-1} \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1}, \end{cases}$$

where

$$\mu_{i,\beta} := \gamma' \left(\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_{i,\beta}|^2 + \frac{1}{2}\beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \right).$$

The equation for $u_{i,\beta}$ is nothing but (3.1): indeed, thanks to the symmetries in $\hat{H}(\mathcal{G}, h)$ (see Remark 1.2), we have $\mu_{i,\beta} = \mu_{j,\beta}$ and $\lambda_{i,\beta} = \lambda_{j,\beta}$ for every $i \neq j$. Finally, $\ell_\beta > 0$ since otherwise $\mathbf{u}_\beta \equiv \mathbf{0}$, in contradiction with the normalization condition.

As far as ℓ is concerned, we introduce an auxiliary functional $I_\infty : \hat{H}_{(\mathcal{G},h)} \to (0, +\infty]$ defined by

$$I_\infty(\mathbf{u}) := \begin{cases} \frac{1}{k}\sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{n-1}} |\nabla u_i|^2}{\int_{\mathbb{S}^{n-1}} u_i^2}\right) & \text{if } u_i u_j = 0 \text{ a.e. on } \mathbb{S}^{n-1} \text{ for any } i \neq j \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that I_β is increasing in β and converges pointwise to I_∞, implying that I_∞ is a weakly lower semi-continuous functional in the weakly closed set $\hat{H}_{(\mathcal{G},h)}$, and that I_β Γ-converges to I_∞ in the weak H^1-topology. Moreover, being the family $\{I_\beta\}$ equi-coercive, any sequence $\{\mathbf{u}_\beta\}$ of minimizers for I_β converges to a minimizer $\mathbf{u}$ of I_∞. Finally, by definition, $\ell > \ell_\beta$ for every $\beta > 0$, whence $\ell > 0$ follows. □

Further properties of the sequence $\{\mathbf{u}_\beta\}$ are collected in the next two lemmas.

Lemma 3.2. *The sequence $\{\mathbf{u}_\beta\}$ is uniformly bounded in* $\mathrm{Lip}(\mathbb{S}^{N-1})$. *Moreover, the sequence (λ_β) is bounded.*

Proof. Let $\{\mathbf{u}_\beta\}$ be a sequence of minimizers for ℓ_β satisfying (3.1), weakly converging to a minimizer $\mathbf{u}$ for ℓ. As $I_\beta(\mathbf{u}_\beta) = \ell_\beta \leq \ell$, there exists $C > 0$ such that

$$\int_{\mathbb{S}^{N-1}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \leq C.$$

Moreover, by weak convergence, $\{\mathbf{u}_\beta\}$ is bounded in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. Therefore, testing the first equation in (3.1) against $u_{i,\beta}$, we deduce that $\{\lambda_\beta\}$ is a bounded sequence of positive numbers, and this implies, through a Brezis-Kato argument, that $\{\mathbf{u}_\beta\}$ is uniformly bounded in $L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^k)$. By the main results in [9], we infer that $\{\mathbf{u}_\beta\}$ is uniformly bounded in $\mathrm{Lip}(\mathbb{S}^{N-1})$ (it is worth to mention that the results in [9] are proved for the Laplace operator in the interior of subsets of $\mathbb{R}^N$, but since they are local, the sphere has no boundary, and the canonical metric on $\mathbb{S}^{N-1}$ is locally equivalent to the Euclidean one, they can be applied also in the present setting). □

Lemma 3.3. *We have $\mathbf{u}_\beta \to \varphi$ strongly in $H^1(\mathbb{S}^{N-1})$ topology, in $\mathcal{C}^{0,\alpha}(\mathbb{S}^{N-1})$ for every $0 < \alpha < 1$, and*

$$\lim_{\beta\to+\infty} \beta \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 u_{j,\beta}^2 = 0.$$

Moreover $\lambda_\beta \to \ell(\ell + N - 2)$, and

$$\begin{cases} -\Delta_\theta \varphi_i = \ell(\ell+N-2)\varphi_i & \text{in } \{\varphi_i > 0\} \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

Proof. Thanks to Lemma 3.2, we can simply apply Theorem 1.4 in [7]. To check that $\lambda_\beta \to \ell(\ell+N-2)$, we observe that by boundedness $\lambda_\beta \to \lambda_\infty \geq 0$ as $\beta \to +\infty$. Therefore, recalling that $\mathbf{u}_\beta \rightharpoonup \varphi$ in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^K)$, for $i = 1, \dots, k$ we have

$$\begin{cases} -\Delta_\theta \varphi_i = \lambda_\infty \varphi_i & \text{in } \{\varphi_i > 0\} \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

This implies that

$$\begin{aligned} \ell &= \frac{1}{k} \sum_i \sqrt{\left(\frac{N-2}{2}\right)^2 + \int_{\mathbb{S}^{N-1}} |\nabla_\theta \varphi_i|^2} - \frac{N-2}{2} \\ &= \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\infty} - \frac{N-2}{2}, \end{aligned}$$

whence the thesis follows. □

The following result is the counterpart of Lemma 4.2 in [13] in a $(\mathcal{G}, h)$-equivariant setting, see also Theorem 5.6 in [2] for an analogue statement in dimension $N = 2$.

Lemma 3.4. *There exists a constant $C > 0$ such that*

$$\ell_\beta \geq \ell - C\beta^{-1/4}.$$

Before proving the lemma, we need a technical result. We recall that $\hat{H}_{(\mathcal{G},h)}$ denotes the set of $(\mathcal{G}, h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$.

Lemma 3.5. *Let $\mathbf{u} \in \hat{H}_{(\mathcal{G},h)}$. Then also the function $\hat{\mathbf{u}}$, defined by*

$$\hat{u}_i = v_i^+ := \left(u_i - \sum_{j \neq i} u_j,\right)^+,$$

belongs to $\hat{H}_{(\mathcal{G},h)}$.

Proof. As $u_i \in H^1(\mathbb{S}^{N-1})$, it follows straightforwardly that $\hat{\mathbf{u}} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. We have to show that it is also $(\mathcal{G}, h)$-equivariant, and to this aim it is sufficient to show that $\mathbf{v}$ is $(\mathcal{G}, h)$-equivariant. This can be checked directly:

$$\begin{aligned} v_{(h(g))^{-1}(i)}(g(x)) &= u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq (h(g))^{-1}(i)} u_j(g(x)) \\ &= u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq i} u_{(h(g))^{-1}(j)}(g(x)) = v_i(x), \end{aligned}$$

where the last equality follows by the fact that $\mathbf{u}$ is $(\mathcal{G}, h)$-equivariant. □

Proof of Lemma 3.4. In order to simplify the notation, only in this proof we write ∇ and Δ instead of ∇_θ and Δ_θ, respectively. Let us consider the functions $\hat{\mathbf{u}}_\beta$, defined in Lemma 3.5. Since the components of $\hat{\mathbf{u}}_\beta$ have disjoint supports, we can use it as competitor for ℓ. We aim at showing that $\hat{\mathbf{u}}_\beta$ is actually close enough to $\mathbf{u}_\beta$ in the energy sense, and in doing this we shall use many times the properties proved in Lemma 3.2. To be precise, we shall prove that there exists a constant $C > 0$ such that

$$1 - C\beta^{-1/2} \leq \int_{\mathbb{S}^{n-1}} \hat{u}_{i,\beta}^2 \leq 1 + C\beta^{-1/2}, \tag{3.2}$$

$$\int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}. \tag{3.3}$$

Before we continue, let us point out that second estimate can be derived from an analogous one, stated as follow: there exists $C > 0$ independent of β and $\bar{\delta} > 0$ such that for almost any $\delta \in (0, \bar{\delta})$ we have

$$\int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta.$$

Indeed, if the previous estimate is satisfied,

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 &= \int_{\{\hat{u}_{i,\beta} > 0\}} |\nabla \hat{u}_{i,\beta}|^2 = \lim_{\delta \to 0^+} \int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \\ &\leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}. \end{aligned}$$

Notice that in principle the value $\bar{\delta}$ could depend on β, but this is not a problem since C is, on the contrary, a universal constant.

Pointwise bounds. The boundedness of $\{\mathbf{u}_\beta\}$ in $\mathrm{Lip}(\mathbb{S}^{N-1})$, Lemma 3.2, implies that there exists a constant $C_1 > 0$ such that

$$\beta^{1/2} u_{i,\beta} u_{j,\beta} \leq C_1 \qquad \forall i \neq j. \tag{3.4}$$

The proof is a straightforward adaptation of the one in [11, Theorem 1.1], which regards the same estimate in subsets of $\mathbb{R}^N$.

As a consequence we have that for each $\theta \in \mathbb{S}^{N-1}$ and each $\beta > 0$

$$\text{there exists at most one index } i \text{ such that } u_{i,\beta}(\theta) \geq 2kC_1^{1/2}\beta^{-1/4}, \tag{3.5}$$

where C_1 is the same constant appearing in (3.4). Indeed, assuming the contrary, there would exists two distinct indices $i \neq j$ satisfying the previous inequality, and hence

$$4k^2 C_1 \beta^{-1/2} \leq u_{i,\beta}(\theta) u_{j,\beta}(\theta) \leq C_1 \beta^{-1/2},$$

a contradiction.

Finally, we observe that

$$\text{if } \hat{u}_{i,\beta}(\theta) = 0, \text{ then } u_{i,\beta}(\theta) \leq 2k(k-1)C_1^{1/2}\beta^{-1/4}. \tag{3.6}$$

If not, we have that (3.5) holds for i, and moreover

$$2k(k-1)C_1^{1/2}\beta^{-1/4} \leq u_{i,\beta}(\theta) \leq \sum_{j \neq i} u_{j,\beta}(\theta) \leq (k-1) \max_{j \neq i} u_{j,\beta}(\theta);$$

hence there exist two indexes for which (3.5) is satisfied in θ, a contradiction.

Integrals bounds for the Laplacian. We prove that there exists a constant $C > 0$ (independent of β) such that

$$\int_{\mathbb{S}^{N-1}} |\Delta u_{i,\beta}| \leq C. \tag{3.7}$$

Indeed, directly form the equation and the divergence theorem

$$0=\int_{\mathbb{S}^{N-1}}(-\Delta u_{i,\beta})=\int_{\mathbb{S}^{N-1}}\lambda_\beta u_{i,\beta}-\beta u_{i,\beta}\sum_{j\neq i}u_j^2;$$

that is

$$0\le\int_{\mathbb{S}^{N-1}}\beta u_{i,\beta}\sum_{j\neq i}u_{j,\beta}^2=\int_{\mathbb{S}^{N-1}}\lambda_\beta u_{i,\beta}\le C,$$

as the functions $u_{i,\beta}$ are boudned in $L^\infty(\mathbb{S}^{N-1})$, and $\{\lambda_\beta\}$ is bounded. Consequently

$$\int_{\mathbb{S}^{N-1}}|\Delta u_{i,\beta}|\le\int_{\mathbb{S}^{N-1}}\lambda_\beta u_{i,\beta}+\beta u_{i,\beta}\sum_{j\neq i}u_{j,\beta}^2\le C.$$

Integrals bounds for the competition term. Using (3.5) and the computations in the previous point, we deduce that

$$\begin{aligned}\int_{\mathbb{S}^{N-1}}\beta\sum_{i\neq j}u_{i,\beta}^2u_{j,\beta}^2&\le\sum_{i\neq j}\Bigg(\|u_{i,\beta}\|_{L^\infty(\{u_{i,\beta}\le u_{j,\beta}\})}\int_{\{u_{i,\beta}\le u_{j,\beta}\}}\beta u_{i,\beta}u_{j,\beta}^2\\&\qquad+\|u_{j,\beta}\|_{L^\infty(\{u_{j,\beta}<u_{i,\beta}\})}\int_{\{u_{j,\beta}<u_{i,\beta}\}}\beta u_{j,\beta}u_{i,\beta}^2\Bigg)\\&\le C\beta^{-1/4}\sum_{i=1}^k\int_{\{u_{i,\beta}\le u_{j,\beta}\}}\beta u_{i,\beta}\sum_{j\neq i}u_{j,\beta}^2\le C\beta^{-1/4}.\end{aligned}$$

Integrals bounds for the normal derivatives. For analogous reasons, we can show that there exists a constant $C>0$ and $\bar\delta>0$ small enough such that for almost every $\delta\in(0,\bar\delta)$ it holds

$$\int_{\partial\{\hat u_{i,\beta}>\delta\}}|\partial_\nu\hat u_{i,\beta}|\le C.$$

Firstly, since for β fixed the function $\hat u_{i,\beta}$ is regular, the set $\partial\{\hat u_{i,\beta}>\delta\}$ is regular for almost every $\delta>0$, by Sard's Lemma. Moreover, since $\hat u_{i,\beta}$ is nonnegative and regular, if $\delta<\bar\delta$ is small enough

$$\int_{\partial\{\hat u_{i,\beta}>\delta\}}|\partial_\nu\hat u_{i,\beta}|=-\int_{\partial\{\hat u_{i,\beta}>\delta\}}\partial_\nu\hat u_{i,\beta}.\tag{3.8}$$

Hence for almost every $\delta\in(0,\bar\delta)$ the set $\partial\{\hat u_{i,\beta}>\delta\}$ is regular, and (3.8) holds. With this choice we are in position to apply the divergence theorem, and consequently

$$\left|\int_{\partial\{\hat u_{i,\beta}>\delta\}}\partial_\nu\hat u_{i,\beta}\right|=\left|\int_{\{\hat u_{i,\beta}>\delta\}}\Delta\hat u_{i,\beta}\right|\le\int_{\{\hat u_{i,\beta}>\delta\}}\sum_{j=1}^k|\Delta u_{j,\beta}|\le C,$$

where C is independent of β by (3.7). With similar computations we have also the uniform estimate

$$\left|\int_{\partial\{\hat u_{i,\beta}>\delta\}}\partial_\nu u_{i,\beta}\right|\le C.$$

Estimates for the $L^2(\mathbb{S}^{N-1})$ norm. Thanks to (3.5) and (3.6), we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}(\hat{u}_{i,\beta}-u_{i,\beta})^2 &= \int_{\{\hat{u}_{i,\beta}>0\}}(\hat{u}_{i,\beta}-u_{i,\beta})^2+\int_{\{\hat{u}_{i,\beta}=0\}}(\hat{u}_{i,\beta}-u_{i,\beta})^2\\ &= \int_{\{u_{i,\beta}>\sum_{j\neq i}u_{j,\beta}\}}\left(\sum_{j\neq i}u_{j,\beta}\right)^2+\int_{\{\hat{u}_{i,\beta}=0\}}u_{i,\beta}^2\leq C\beta^{-1/2},\end{aligned}$$

whence (3.2) follows.

Estimates for the $H^1(\mathbb{S}^{N-1})$ seminorm. As a last step, we wish to estimate the L^2 norm of $\nabla\hat{u}_{i,\beta}$. Since $\partial\{\hat{u}_{i,\beta}>\delta\}$ is regular, we can apply the divergence theorem deducing that

$$\begin{aligned}\int_{\{\hat{u}_{i,\beta}>\delta\}}|\nabla\hat{u}_{i,\beta}|^2 &= \int_{\{\hat{u}_{i,\beta}>\delta\}}(-\Delta\hat{u}_{i,\beta})\hat{u}_{i,\beta}+\int_{\partial\{\hat{u}_{i,\beta}>\delta\}}(\partial_\nu\hat{u}_{i,\beta})\hat{u}_{i,\beta}\\ &= \underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}}(-\Delta u_{i,\beta})u_{i,\beta}}_{\text{(I)}}+\int_{\{\hat{u}_{i,\beta}>\delta\}}\Delta u_{i,\beta}\sum_{j\neq i}u_{j,\beta}\\ &\quad+\underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}}\Delta\left(\sum_{j\neq i}u_{j,\beta}\right)\hat{u}_{i,\beta}}_{\text{(II)}}+\delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}}\partial_\nu\hat{u}_{i,\beta}\end{aligned}$$

The first term (I) can be bounded using the equation:

$$\begin{aligned}\int_{\{\hat{u}_{i,\beta}>\delta\}}(-\Delta u_{i,\beta})u_{i,\beta} &= \int_{\{\hat{u}_{i,\beta}>\delta\}}\lambda_\beta u_{i,\beta}^2-\beta u_{i,\beta}^2\sum_{j\neq i}u_{j,\beta}^2\\ &\leq \int_{\mathbb{S}^{N-1}}\lambda_\beta u_{i,\beta}^2-\beta u_{i,\beta}^2\sum_{j\neq i}u_{j,\beta}^2+\int_{\mathbb{S}^{N-1}\setminus\{\hat{u}_{i,\beta}>\delta\}}\beta u_{i,\beta}^2\sum_{j\neq i}u_{j,\beta}^2\\ &= \int_{\mathbb{S}^{N-1}}|\nabla u_{i,\beta}|^2+\int_{\mathbb{S}^{N-1}\setminus\{\hat{u}_{i,\beta}>\delta\}}\beta u_{i,\beta}^2\sum_{j\neq i}u_{j,\beta}^2.\end{aligned}$$

The term (II) can be expanded further as

$$\begin{aligned}\int_{\{\hat{u}_{i,\beta}>\delta\}}\Delta\left(\sum_{j\neq i}u_{j,\beta}\right)\hat{u}_{i,\beta} &= -\int_{\{\hat{u}_{i,\beta}>\delta\}}\nabla\left(\sum_{j\neq i}u_{j,\beta}\right)\cdot\nabla\hat{u}_{i,\beta}\\ &\quad+\delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}}\partial_\nu\left(\sum_{j\neq i}u_{j,\beta}\right)=\int_{\{\hat{u}_{i,\beta}>\delta\}}\left(\sum_{j\neq i}u_{j,\beta}\right)\Delta\hat{u}_{i,\beta}\\ &\quad-\int_{\partial\{\hat{u}_{i,\beta}>\delta\}}\left(\sum_{j\neq i}u_{j,\beta}\right)\partial_\nu\hat{u}_{i,\beta}+\delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}}\partial_\nu\left(\sum_{j\neq i}u_{j,\beta}\right).\end{aligned}$$

Recollecting the previous computations, and using again (3.5), we have

$$\begin{aligned}\int_{\{\hat u_{i,\beta}>\delta\}} |\nabla \hat u_{i,\beta}|^2 &\le \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat u_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \\ &\quad + \int_{\{\hat u_{i,\beta}>\delta\}} \Delta u_{i,\beta} \sum_{j\neq i} u_{j,\beta} + \int_{\{\hat u_{i,\beta}>\delta\}} \left(\sum_{j\neq i} u_{j,\beta}\right) \Delta \hat u_{i,\beta} \\ &\quad - \int_{\partial\{\hat u_{i,\beta}>\delta\}} \left(\sum_{j\neq i} u_{j,\beta}\right) \partial_\nu \hat u_{i,\beta} + \delta \int_{\partial\{\hat u_{i,\beta}>\delta\}} \partial_\nu u_{i,\beta} \\ &\le \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta,\end{aligned}$$

which, as already observed, implies (3.3).

With (3.2) and (3.3) we are in position to complete the proof. By minimality $\ell \le I_\infty(\hat{\mathbf{u}}_\beta)$ for every β, which gives

$$\begin{aligned}\ell &\le \frac{1}{k}\sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}} |\nabla \hat u_{i,\beta}|^2}{\int_{\mathbb{S}^{N-1}} \hat u_{i,\beta}^2}\right) \le \frac{1}{k}\sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}}{1 - C\beta^{-1/2}}\right) \\ &\le \frac{1}{k}\sum_{i=1}^k \gamma\left(\int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \frac{1}{2}\beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2\right) + C\beta^{-1/4} = \ell_\beta + C\beta^{-1/4}. \qquad \square\end{aligned}$$

The proof of Proposition 1.5 can be obtained in a somehow usual way.

Sketch of the proof of Proposition 1.5. Arguing as in Section 7 of [3], or [7, Lemma 2.5], or else [9, Theorem 3.14], it is possible to check that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right) = -\frac{2k\ell}{r} + \frac{2}{r}\sum_i \gamma\left(\frac{r^2\int_{\partial B_r} |\nabla u_i|^2 + \frac{1}{2}u_i^2\sum_{j\neq i} u_j^2}{\int_{\partial B_r} u_i^2}\right).$$

Changing variables in the integrals (see Theorem 3.14 in [9] for the details), we deduce that

$$\sum_i \gamma\left(\frac{r^2\int_{\partial B_r} |\nabla u_i|^2 + \frac{1}{2}u_i^2\sum_{j\neq i} u_j^2}{\int_{\partial B_r} u_i^2}\right) \ge k\ell_{r^2},$$

where ℓ_{r^2} denotes the optimal value ℓ_β for $\beta = r^2$. Coming back to the previous equation, and using Lemma 3.4, we conclude that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right) \ge \frac{2k}{r}(\ell_{r^2} - \ell) \ge -2kCr^{-3/2},$$

and integrating the thesis follows. $\square$

4. Construction of equivariant solutions

For an admissible triplet $(k, \mathcal{G}, h)$, we prove the existence of a $(\mathcal{G}, h)$-equivariant solution to (1.1) with k components. We partially follow the method introduced in [2], which consists in two steps:

- firstly, we prove the existence of a sequence of $(\mathcal{G}, h)$-equivariant solutions $\mathbf{V}_R$, defined in balls of increasing radii $R \to +\infty$;

- secondly, we show that such sequence converges locally uniformly in $\mathbb{R}^N$ to a nontrivial solution.

With respect to [2], we modify the construction conveniently choosing R from the beginning; this simplifies substantially the proof of the convergence of $\{\mathbf{V}_R\}$, and we refer to the forthcoming Remark 4.4 for more details. Finally, in the last part of the proof we characterize the growth of the solution using the Alt-Caffarelli-Friedman monotonicity formula for $(\mathcal{G}, h)$-equivariant solutions.

By Lemma 3.1, we know that the optimal value ℓ (see Definition 1.4) is achieved by a nonnegative $(\mathcal{G}, h)$-equivariant function $\varphi \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. Differently from the previous section, we suppose that

$$\sum_{i=1}^{k} \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1 \quad \Longleftrightarrow \quad \int_{\mathbb{S}^{N-1}} \varphi_i^2 = \frac{1}{k}. \tag{4.1}$$

This choice is possible, since the minimum problem for ℓ is invariant under scaling of type $t \mapsto t\varphi$ with $t \in \mathbb{R}$, and simplifies some computation.

Lemma 4.1. *For any $\beta > 0$ there exists a $(\mathcal{G}, h)$-equivariant solution $\{\mathbf{U}_\beta\}$ to the problem*

$$\begin{cases} \Delta U_{i,\beta} = \beta U_{i,\beta} \sum_{j \neq i} U_{j,\beta}^2 & \text{in } B_1 \\ U_{i,\beta} > 0 & \text{in } B_1 \\ U_{i,\beta} = \varphi_i & \text{on } \partial B_1 = \mathbb{S}^{N-1}. \end{cases}$$

Moreover

(i) $U_{i,\beta}(0) = U_{j,\beta}(0)$ $\forall i, j = 1, \dots, k$ *and* $\beta > 0$;

(ii) *letting*

$$\mathcal{E}_\beta(\mathbf{U}) = \int_{B_1} \sum_{i=1}^{k} |\nabla U_i|^2 + \beta \sum_{i<j} U_i^2 U_j^2,$$

the uniform estimate $\mathcal{E}_\beta(\mathbf{U}_\beta) \le \ell$ holds.

(iii) *there exists a Lipschitz continuous function $\mathbf{0} \not\equiv \mathbf{U}_\infty$ such that, up to a subsequence, $\mathbf{U}_\beta \to \mathbf{U}_\infty$ in $\mathcal{C}^{0,\alpha}(B_1)$ and in $H^1_{\mathrm{loc}}(B_1)$.*

Proof. It is not difficult to check that the functional $\mathcal{E}_\beta$ admits a minimizer $\mathbf{U}_\beta$ in the H^1-weakly closed set of the $(\mathcal{G}, h)$-equivariant functions in $H^1(B_1, \mathbb{R}^k)$ with the prescribed boundary conditions. The fact that such minimizer solves the Euler-Lagrange equation is a consequence of Palais' principle of symmetric criticality. Property (i) follows straightforwardly by the equivariance. Concerning property (ii), we introduce the ℓ-homogeneous extension of φ, defined by

$$\phi(x) := |x|^\ell \varphi\left(\frac{x}{|x|}\right).$$

By minimality $\mathcal{E}_\beta(\mathbf{U}_\beta) \le \mathcal{E}_\beta(\phi)$, so that it remains to check that $\mathcal{E}_\beta(\phi) \le \ell$. At first, since φ_i is an eigenfunction of $-\Delta_\theta$ on $\{\varphi_i > 0\}$ associated to the eigenvalue $\ell(\ell + N - 2)$, the function ϕ_i is harmonic in $\{\phi_i > 0\}$. Furthermore, by definition,

$$\sum_i \int_{\partial B_1} \phi_i^2 = 1$$

for every i. Therefore, using the Euler formula for homogeneous functions, we deduce that

$$\begin{aligned}\mathcal{E}_\beta(\phi) &= \sum_i \int_{B_1} |\nabla \phi_i|^2 = \sum_i \int_{\{\phi_i>0\}} |\nabla \phi_i|^2 \\ &= \sum_i \int_{\partial B_1 \cap \{\phi_i>0\}} \phi_i \partial_\nu \phi_i = \ell \sum_i \int_{\partial B_1 \cap \{\phi_i>0\}} \phi_i^2 = \ell.\end{aligned}$$

It remains to prove (iii). By (ii) and the boundary conditions, the sequence $\{\mathbf{U}_\beta\}$ is bounded in $H^1(B_1)$, and hence it converges weakly to some limit $\mathbf{U}_\infty$. By compactness of the trace operator, $\mathbf{U}_\infty \not\equiv \mathbf{0}$. All the functions $\mathbf{U}_\beta$ are nonnegative, subharmonic and have the same boundary conditions, and hence by the maximum principle they are uniformly bounded in $L^\infty(B_1)$. This, as recalled in Section 2, implies the thesis. □

We plan to use the solutions of Lemma 4.1 in order to construct entire solutions to (1.1). Our method is based on a simple blow up argument. For a positive radius r_β to be determined, we introduce

$$V_{i,\beta}(x) := \beta^{1/2} r_\beta U_{i,\beta}(r_\beta x).$$

By definition, $\mathbf{V}_\beta$ solves

$$\Delta V_{i,\beta} = V_{i,\beta} \sum_{j \neq i} V_{j,\beta}^2 \qquad \text{in } B_{1/r_\beta}.$$

A convenient choice of r_β is suggested by the following statement.

Lemma 4.2. *For any fixed $\beta > 1$ there exists a unique $r_\beta > 0$ such that*

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1.$$

Moreover $r_\beta \to 0$, and consequently $B_{1/r_\beta} \to \mathbb{R}^N$, in the sense that for any compact $K \subset \mathbb{R}^N$ it results $K \Subset B_{1/r_\beta}$ provided β is sufficiently large.

Proof. We have to find $r_\beta > 0$ such that $\beta r^2 H(\mathbf{U}_\beta, r_\beta) = 1$. The strict monotonicity of $H(\mathbf{U}_\beta, \cdot)$ (see Section 2) implies the strict monotonicity of the continuous function $r \mapsto \beta r^2 H(\mathbf{U}_\beta, r)$. By regularity, for any β fixed

$$\lim_{r\to 0} \beta r^2 H(\mathbf{U}_\beta, r) = \lim_{r \to 0} \beta \frac{r^2}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k U_{i,\beta}^2 = \beta \lim_{r\to 0} r^2 \cdot \sum_{i=1}^k U_{i,\beta}^2(0) = 0,$$

and by the normalization (4.1) it results $\beta H(\mathbf{U}_\beta, 1) = \beta > 1$. This proves existence and uniqueness of r_β. If by contradiction $r_\beta \geq \bar{r} > 0$, then by Lemma 4.1-(iii) and by the monotonicity of $H(\mathbf{U}_\beta, \cdot)$ we would have

$$1 = \beta r_\beta^2 H(\mathbf{U}_\beta, r_\beta) \geq \frac{\beta \bar{r}^2}{2} \frac{1}{\bar{r}^{N-1}} \int_{\partial B_{\bar{r}}} \sum_{i=1}^k U_{i,\infty} \geq \beta C,$$

which gives a contradiction for $\beta > 1/C$. In order to bound from below the second to last term, we recall that since $\mathbf{0} \not\equiv \mathbf{U}_\infty$, we have $H(\mathbf{U}_\infty, r) \neq 0$ for all $0 < r < 1$ (see Section 2). □

Lemma 4.3. *Up to a subsequence,* $\mathbf{V}_\beta \to \mathbf{V}$ *in* $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$*, and* $\mathbf{V}$ *is an entire* $(\mathcal{G}, h)$*-equivariant solution of* (1.1) *with* $N(\mathbf{V}, r) \le \ell$ *for every* $r > 0$*.*

Proof. Since $\mathcal{E}_\beta(\mathbf{U}_\beta) \le \ell$ and $H(\mathbf{U}_\beta, 1) = 1$, by scaling and using the monotonicity of the Almgren quotient we have

$$N(\mathbf{V}_\beta, r) \le N\left(\mathbf{V}_\beta, \frac{1}{r_\beta}\right) = N(\mathbf{U}_\beta, 1) \le \frac{\mathcal{E}(\mathbf{U}_\beta)}{H(\mathbf{U}_\beta, 1)} \le \ell \tag{4.2}$$

for every $0 < r < 1/r_\beta$, $\beta > 0$. Let now $r > 0$; then for β sufficiently large

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} \log H(\mathbf{V}_\beta, r) &= \frac{2}{r} N_\beta(\mathbf{v}_\beta, r) + \frac{2}{r^{N-1} H(\mathbf{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2 \\ &\le \frac{2\ell}{r} + \frac{2}{r^{N-1} H(\mathbf{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2.\end{aligned}$$

Integrating the inequality with $r \in (1, R)$, and recalling (2.2), we infer that

$$\frac{H(\mathbf{V}_\beta, R)}{R^{2\ell}} \le H(\mathbf{V}_\beta, 1) e^\ell = e^\ell \quad \forall R \ge 1, \tag{4.3}$$

independently of β. By subharmonicity and standard elliptic estimates, we deduce that $\mathbf{V}_\beta$ converges in $\mathcal{C}^2(B_R)$ to some limit $\mathbf{V}^R$, and since R has been arbitrarily chosen, a diagonal selection gives convergence to an entire limit $\mathbf{V}$, which is clearly $(\mathcal{G}, h)$-equivariant. Since $\mathbf{V}$ solves (1.1) and

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1 \quad \text{and } V_{i,\beta}(0) = V_{j,\beta}(0) \text{ for all } i, j$$

(see Lemmas 4.1 and 4.2), all the components of $\mathbf{V}$ are nontrivial, and hence non-constant. □

We now show that the growth rate of the solution is exactly equal to ℓ. In light of the upper bound on the Almgren quotient proved in the previous lemma, this is a consequence of Proposition 1.4.

Proof of Proposition 1.4. Let us assume that for a $(\mathcal{G}, h)$-equivariant solution it results $N(\mathbf{V}, r) \le \ell - \varepsilon$ for every $r > 0$. We consider the blow-down sequence

$$\mathbf{V}_R(x) = \frac{1}{\sqrt{H(\mathbf{V}, R)}} \mathbf{V}(Rx).$$

By Theorem 1.4 in [8], it converges in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ to a limit $\mathbf{W}$, which is moreover segregated, nonnegative, homogeneous with homogeneity degree $\delta \le \ell - \varepsilon$, and such that $\Delta W_i = 0$ in $\{W_i > 0\}$. The uniform convergence entails the $(\mathcal{G}, h)$-equivariance, and hence the trace $\hat{\mathbf{w}}$ of $\mathbf{W}$ on the sphere $\mathbb{S}^{N-1}$ is an admissible competitor for ℓ, in the sense that $\ell \le I_\infty(\hat{\mathbf{w}})$ (we recall that I_∞ is defined in Lemma 3.1). The value $I_\infty(\hat{\mathbf{w}})$ can be computed explicitly: indeed, by harmonicity, homogeneity and symmetry, $\hat{w}_i$ is an eigenfunction of the Laplace-Beltrami operator $-\Delta_\theta$ on a subdomain of $\mathbb{S}^{N-1}$, associated to the eigenvalue $\delta(\delta + N - 2)$. This, by definition, implies that $I_\infty(\hat{\mathbf{w}}) = \delta < \ell$, in contradiction with the minimality of ℓ. □

So far we proved the existence of a $(\mathcal{G}, h)$-equivariant solution having growth rate ℓ in the weak sense of (2.3). It remains to show that the stronger condition (1.5) holds. Before, we make the following remark.

Remark 4.4. Both Theorem 1.3 and [2, Theorem 1.6] are based upon the same two-steps procedure: construction of solutions in balls B_R of increasing radius, and passage to the limit as $R \to +\infty$. The main difference stays in the fact that while in [2] the authors prescribed the value of the functions on the boundary ∂B_R, we prescribed the value on ∂B_1, conveniently choosing r_β. This permits to simplify very much the proof of the convergence, since by the doubling property (4.3), the normalization on ∂B_1 is enough to have $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$ convergence of our approximating sequence. In [2, page 123], such compactness is proved in a different way, using fine tools such as Proposition 5.7 therein, which seems difficult to generalize in higher dimension.

Lemma 4.5. *It holds*

$$\lim_{r\to\infty} \frac{1}{r^{2\ell}} H(\mathbf{V}, r) \in (0, +\infty).$$

Proof. It is easy to prove that the limit exists and it is less than 1. Indeed

$$\frac{d}{dr} \log \frac{H(\mathbf{V}, r)}{r^{2\ell}} = \frac{H'(\mathbf{V}, r)}{H(\mathbf{V}, r)} - \frac{2\ell}{r} = \frac{2}{r}(N(\mathbf{V}, r) - \ell) \le 0,$$

and by construction $H(\mathbf{V}, 1) = 1$. Letting

$$L = \lim_{r\to\infty} \frac{H(\mathbf{V}, r)}{r^{2\ell}}$$

we are left to show that $L > 0$. Recalling that $N(\mathbf{V}, +\infty) = \ell$, we have

$$L = \lim_{r\to\infty} \left(\frac{E(\mathbf{V}, r)}{r^{2\ell}} \right) \cdot \lim_{r\to+\infty} \frac{H(\mathbf{V}, r)}{E(\mathbf{V}, r)} \ge \frac{1}{\ell} \liminf_{r\to\infty} \frac{E(\mathbf{V}, r)}{r^{2\ell}},$$

and the thesis follows if

$$\liminf_{r\to\infty} \frac{E(\mathbf{V}, r) + H(\mathbf{V}, r)}{r^{2\ell}} > 0.$$

To this aim, we note that with computations analogue to those in [11, Conclusion of the proof of Theorem 1.5] we can prove that

$$\frac{E(\mathbf{V}, r) + H(\mathbf{V}, r)}{r^{2\ell}} \ge \frac{C}{r^{2\ell}} \left(J_1(r) \dots J_k(r) \right)^{1/k} = C \left(\frac{1}{r^{2\ell k}} J_1(r) \dots J_k(r) \right)^{1/k},$$

where the integrals J_i are evaluated for the function $\mathbf{V}$. Since $\mathbf{V}$ is a $(\mathcal{G}, h)$-equivariant solution of (1.1), we are in position to apply the Alt-Caffarelli-Friedman monotonicity formula of Proposition 1.5, whence

$$\frac{E(\mathbf{V}, r) + H(\mathbf{V}, r)}{r^{2\ell}} \ge C \left(J_1(1) \dots J_k(1) \right)^{1/k} e^{Cr^{-1/2}} \ge C e^{Cr^{-1/2}}$$

for every $r > 1$. $\square$

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