Bubbles in assets with finite life*

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Abstract

We study the speculative value of a finitely lived asset when investors disagree and short sales are limited. In this case, investors are willing to pay a speculative value for the resale option they obtain when they acquire the asset. We characterize the equilibrium speculative value as a solution to a fixed point problem for a monotone operator $\mathbb{F}$. A *Dynamic Programming Principle* applies and is used to show that the minimal solution to the fixed-point problem is a *viscosity solution* of a naturally associated (non-local) obstacle problem. This obstacle type free boundary problem was the focus of Berestycki et al. (2014), who proved a comparison principle and existence and uniqueness of the viscosity solution. Combining the monotonicity of the operator $\mathbb{F}$ and the comparison principle we obtain several comparison of solution results. We also use the characterization of the exercise boundary of the obstacle problem to study the effect of an increase in the costs of transactions on the value of the bubble and on the volume of trade, and in particular to quantify the effect of a small transaction (Tobin) tax.
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1 Introduction

Asset price bubbles are episodes in which asset prices seem to vastly exceed fundamentals. Well known historical episodes that have been deemed as bubbles include the Dutch Tulip-Mania in the 17th century, the South-Sea Bubble in the 18th century, the Railway-Mania in 19th century Britain, the Roaring-Twenties in the 20th century, the Internet or dot.com, and the recent Credit bubble. One striking observation is that bubble episodes are often accompanied by trading frenzies.[1]

Economists have written many theoretical models of equilibria in which divergence between fundamental valuations and market prices can be in principle observed. One standard class of models shows that bubbles can actually survive when agents have rational expectations and are symmetrically informed. In these models, agents pay more for an asset than the value of its expected discounted future dividends, because they hope to sell the asset for an even higher price in the future, at least with some probability. Asset bubbles can actually collapse, but they cannot be expected to implode for sure. Hence these models must assume that assets are, at least in principle, infinitely lived. In addition, rational-bubbles models are incapable of explaining the observed correlation between bubbles and trading.

An alternative class of models for asset-pricing bubbles combines agents with fluctuating heterogeneous beliefs and an assumption that it is more expensive to go short an asset than going long that same asset.[2] An observation that goes back to Miller (1977) in a static context is that in the presence of these cost asymmetries, the view of optimists - natural buyers - is expressed more fully than that of pessimists - natural sellers. Even if beliefs are on average unbiased, prices would be biased. Fluctuating beliefs

[1]See Carlos et al. (2010) for the increase in trading volume during the South Sea Bubble, Hong and Stein (2007) for the *Roaring Twenties*, Cochrane (2002), Lamont and Thaler (2003) to Ofek and Richardson (2003) for the internet bubble, and Xiong and Yu (2011) for evidence concerning a recent Chinese warrants bubble

[2]These two basic assumptions - heterogeneous beliefs and higher costs of going short - are far from standard in the asset-pricing literature. The existence of differences in beliefs is obvious for the vast majority of market practitioners, but economists have produced a myriad of results showing that "rational" investors cannot agree to disagree. Similarly, there are good economic reasons why investors should have more difficulty going short than going long, but most economic models assume no asymmetry.

give current optimists the hope that an even more optimistic buyer may appear in the future. Hence buyers would be willing to pay more than the value they attribute to an assets' future payoffs, because the ownership of the asset gives her the option to resell the asset to a future optimist.[3]

Scheinkman and Xiong (2003), inspired by a pioneer paper by Harrison and Kreps (1978), developed a fully dynamic continuous time equilibrium model of asset price bubbles in which heterogeneous beliefs are generated because agents disagree on the precision of the information they observed.[4] Scheinkman and Xiong (2003) studied a market for a single asset in limited supply and many risk-neutral agents facing proportional trading costs. For technical reasons, they ruled out short-sales although their qualitative results would survive if instead they had only assumed costly short-sales. There are two groups of agents and traders in each group attribute excessive precision to a different set of signals. As the information on the signals flows, the group of agents that places excessive confidence on the value of a particular signal would overreact to the realizations of that signal. In this way, the group of traders that is relatively optimistic now may become relatively pessimistic in the future. In a linear-Gaussian framework the difference in mean beliefs x is the state variable of the model. The buyer of the asset today acquires an option to resell that asset to other more optimistic traders in the future. In equilibrium buyers would be part of the most optimistic group but would be willing to pay in excess of their optimistic views, because they value the option to resell. The value of this option can be legitimately titled a bubble. The resale option is *American*, that is it can be exercised at any time. Thus the value of the resale option is given by an associated optimal stopping time but the value of stopping in turn is given by a stopping time problem faced by the new buyer. Because of this recursive aspect in the option valuation, the value of the option is characterized by a non-local obstacle problem. To exploit stationarity, they assumed that the asset was infinitely lived, and were able to construct an explicit solution to the obstacle problem and show that the optimal stopping was characterized by a level in the difference in (mean) beliefs. In equilibrium, each time the

[3] An alternative mathematical theory of bubbles postulates only no-arbitrage. In the simplest case of an asset that pays only a final dividend and complete markets, the price process of the asset, S_t, is a local martingale under the unique risk-neutral measure Q. The fundamental value of the asset S_t^* is the expected value (under Q) of the payoff, and necessarily a martingale. The difference between S and S^* is non-negative and is defined as the bubble. For a summary and further references see Protter (2012). Because it is a no-arbitrage theory, there are no implications concerning trading volume.

[4] See also Chen and Kohn (2011) and Dumas et al. (2009)

difference in beliefs between the most optimistic group and the group of the current asset holders reaches a critical value, a trade occurs. Scheinkman and Xiong (2003) also showed that as trading costs approaches zero, trading volume goes to infinity and that when trading costs are small, increases on the degree of overconfidence that traders have in their own signal increase the value of the option and the volume of trade.

In this paper we extend the model of Scheinkman and Xiong (2003) to finitely lived assets. This extension makes the model applicable to credit instruments, which are typically finitely lived. A finite horizon model is also needed to explain the bubble on the value of Chinese warrants that was documented by Xiong and Yu (2010).

When assets have a finite life the value of the option $q(x, t)$ depends on the difference in mean beliefs x and time to expiration t. The function q must satisfy a fixed point problem that is a natural finite-horizon version of the problem studied by Scheinkman and Xiong (2003). However, while Scheinkman and Xiong (2003) were able to produce an explicit solution, we must use a monotone fixed point argument, partly adapted from Chen and Kohn (2011), to establish the existence of a solution. As Bouchard (2007) we establish a *Dynamic Programming Principle* and use it to show that the *minimal* solution q to the fixed-point problem[5] must be a *viscosity solution* of a naturally associated (non-local) obstacle problem.[6] We can then exploit results in Berestycki et al. (2014) that guarantee the existence and uniqueness of the viscosity solution to the obstacle problem. Instead of a critical value in the difference in opinions that triggers trades we obtain an exercise boundary $k(t)$ - trading occurs at t if the beliefs of the most optimistic group exceeds that of the current holders by at least $k(t)$. By symmetry, the buyer of the asset has mean beliefs that differ from the mean beliefs of the current holder by $-k(t)$. We define the speculative component (bubble) as the amount that an agent pays in addition to her own valuation, that is exactly $q(-k(t), t)$. Buyers will be part of the most optimistic group and the difference between their valuation and a rational fundamental valuation could also be rightfully claimed as a portion of the bubble. Thus the definition of bubble that we use is somewhat conservative.

The equilibrium solution q to value of the bubble satisfies some natural monotonicity properties; it increases with the time to expiration or with

[5]In Scheinkman and Xiong (2003) the focus is also on a minimum solution, because it minimizes the size of the bubble

[6]Zariphopoulou (1994) and Theorem VIIII.5.1 in Fleming and Soner (1993) are early examples of the use of the dynamic programming principle to establish the necessity of viscosity solutions.

the difference in opinions x. In addition, q is a convex function of x, and as a consequence, an increase in the volatility of the difference in opinions σ increases q. An increase in the proportional transaction costs c or in the risk-free rate r decreases q. Some monotonicity properties are established using the monotonicity of the operator defining the fixed point q; others use a comparison principle established in Berestycki et al. (2014) for the associated obstacle problem. Berestycki et al. (2014) also shows that the exercise boundary k can be obtained from a related local obstacle problem that also satisfies a comparison principle. Their results insure that k is increasing with t and goes to infinity as $t \to T$, that is as the time to expiration converges to zero. The comparison principle allows us to prove monotonicity properties for the exercise boundary k. The exercise boundary is increasing as a function of c, σ or r. Combining the monotonicity results on the value of the resale option q and the exercise boundary k, one can show that the size of the bubble $q(-k(t), t)$ decreases with c or r. Thus a decrease in the cost of transaction or in the risk-free interest rate increase both the volume of trades and the size of the bubble. This shows that interest rates and transaction costs are potential sources for the correlation between bubbles and trading volume that is often observed. As $c \to 0$, $k(t) \to 0$ and thus owners of the asset exercise their option to sell soon after it gets *in the money*. Nonetheless the value of the resale option is maximized, illustrating the local time character of trading in the limit. Although the monotonicity results of q and k do not lead to a straightforward answer on the dependence of the size of the bubble on the volatility of the difference of opinions σ, we are able to show using a rescaling argument that $b(t)$ increases with σ, and thus the size of the bubble increases with the volatility of differences in opinion.

Berestycki et al. (2014) also shows that as $c \to 0$, $k(t, c) \sim c^{1/3}\bar{k}(t)$, and give an explicit formula for $\bar{k}$. We use this result and some formal asymptotics to argue that the elasticity of the median time between trades is $2/3$ at $c = 0$. In addition, we show that the elasticity of the bubble with respect to c goes to zero as $c \to 0$. If one think of a Tobin Tax as an increase in c, these results argue that starting from a scenario of small transaction costs, a Tobin Tax has a large effect on volume, but a much more modest effect on the size of the bubble.

The paper is organized as follows. Section 2 contains a description of the model and a derivation of the stochastic formulation of the equilibrium value of the resale option. The existence of a minimal fixed point to this stochastic formulation is established in Section 3. In Section 4, we give an intuitive derivation of the obstacle problem that the equilibrium value of the option must satisfy and provide statements of the comparison principle and existence theorems that Berestycki et al. (2014) establishes for the obstacle problem. We show in Section 5 that a dynamic programming principle holds and that it implies that the minimal solution to the stochastic formulation is a viscosity solution to the obstacle problem. Section 6 defines the exercise boundary and the bubble. Section 7 contains comparative statics results and in Section 8 we discuss the effect of the trading cost c on the volume of trade and value of the bubble. Section 9 concludes. The Appendices contain some supplementary materials and proofs.

2 Derivation of the model

2.1 Dividends and information

As in Scheinkman and Xiong (2003) we consider two sets of risk-neutral investors A and B. For simplicity we will sometimes refer to an investor in group $C \in \{A, B\}$ as investor C. All investors discount future payoffs at a continuously compounded rate $r > 0$. There is a single risky asset in a fixed supply that we normalize to 1, which provides a flow of dividends up to a maturity $T > 0$. We write D_t for the process of cumulative dividends and since no dividends are paid after the maturity T, $D_t = D_T$ if $t \geq T$.

Investors disagree on the future flow of dividends. We model this by assuming that $\{\Omega, P^C, \mathcal{F}\}$ is a probability space for $C \in \{A, B\}$, with $P^A \sim P^B$. We assume that $(W^C, W^{C,D})$ is a 2-dimensional Brownian motion in $\{\Omega, P^C, \mathcal{F}\}$, such that under beliefs P^C of investors in group $C \in \{A, B\}$ the process of dividends is given by the following pair of diffusions:

$$\begin{aligned} dD_t &= \hat{f}_t^C dt + \sigma_D dW_t^{C,D} && (2.1) \\ d\hat{f}_t^C &= -\lambda(\hat{f}_t^C - \bar{f})dt + \sigma_{\hat{f}}[\varsigma dW^{C,D} + \sqrt{1-\varsigma^2} dW_t^C]. && (2.2) \end{aligned}$$

Here, $0 \leq \varsigma \leq 1$. When $\hat{f}_t^A > \hat{f}_t^B$ the investors in group A are relatively optimistic.

To complete the model we need to consider the views that investors in group $C \in \{A, B\}$ have of the evolution of beliefs of the investors in the complementary group. We assume that investors in both groups know the current value of $\hat{f}_t^A$ and $\hat{f}_t^B$. We write $X_s = \hat{f}_s^B - \hat{f}_s^A$. We assume that all investors forecast that X satisfies:

$$dX_s = -\rho X_s ds + \sigma dW_s \qquad (2.3)$$

for $t \leq s \leq T$, W is a Brownian motion from the point of view of *both* group of investors, and the future (past) increments to W are independent of the past (future) increments to $W^{C,D}$ and W^C for $C \in \{A, B\}$. Assuming that investors agree on the current value of the difference in beliefs and on evolution of differences in beliefs amounts to assuming that investors in one group know the model used by investors in the other group and agree to disagree. The independence of W on past values of the other Brownian will allow us to treat X as a state variable.

Scheinkman and Xiong (2003) derive a particular version of the system of equations (2.1)-(2.3), that satisfy the independence hypothesis on W, by assuming that investors observe two signals s^A and s^B that are correlated with the drift of dividends and that the different groups exaggerate the informativeness of the different signal. The degree by which investors in group C exaggerates the informativeness of signal s^C is characterized by a parameter $\phi \geq 0$. In Scheinkman and Xiong (2003) $\rho > \lambda \geq 0$, and $\sigma > 0$ if $\phi > 0$. As $\phi \to 0$ investors differences of opinion simply reflect their initial priors, $\rho \to \lambda$ and $\sigma \to 0$. Here we opt to go directly to the general model described by (2.1)-(2.3) and concentrate in deriving its implications for finitely lived assets.

In what follows we will assume

Assumption 2.1 *$\sigma > 0$, $\rho > 0$ and $\rho > \lambda > -r$*

Since $\sigma > 0$, differences in beliefs are volatile and may change sign. The assumption $\rho \geq \lambda$ means that differences in beliefs relax towards the origin faster than the drift of dividends relaxes towards $\bar{f}$. Although we are mostly interested in the case $\lambda > 0$, it suffices to assume that $\lambda + r > 0$.[7]

We will write $\mathcal{G}_t$, $0 \leq t \leq T$ for the completed filtration generated by the vector $Z = (D, \hat{f}^A, X)$, or equivalently $\tilde{Z} = (D, \hat{f}^B, X)$. In turn this filtration can be viewed as the completed filtration generated by the initial conditions Z_0 and the triple $(W^{A,D}, W^A, W)$ (or equivalently $(W^{B,D}, W^B, W)$). Since $(D, \hat{f}^A, X)$ is Markov if $\mathbb{E}^C[h]$ exists and $s \geq t$,

$$\mathbb{E}_t^C\left[h(Z_s)\right] := \mathbb{E}^C\left[h(Z_s)|\mathcal{G}_t\right] = E^C\left[h(Z_s)|Z_t\right].$$

Notice that equation (2.2) implies:

$$\mathbb{E}_t^C[\int_t^T e^{-r(u-t)} \hat{f}_u^C du \mid \hat{f}_t^C = y] = \alpha(T-t, r)\bar{f} + \alpha(T-t, r+\lambda)(y - \bar{f}), \tag{2.4}$$

[7] The condition $\lambda + r > 0$ is necessary for the existence of an equilibrium of the associated stationary problem (see Scheinkman and Xiong (2003)).

where

$$\alpha(t,\mu) = \frac{1 - e^{-t\mu}}{\mu}.$$

2.2 Trading

The price of the asset is quoted ex-dividend, therefore the value of the asset after maturity T is zero.[8] We call p_t^A the maximum price an investor in group A is willing to pay for the asset at time t. Since there are no dividends at any time after T, $p_T^A = 0$. We assume a large number of investors in each group bidding for the fixed asset supply, so it is natural to also assume that if an investor in A buys the asset at t he must pay p_t^A. The buyer will receive the dividend of the asset from t up to a time τ, $t < \tau < T$, at which she sells the asset. We assume that there is a fixed cost $c \geq 0$ for any transaction. This means that if investor A pays at t the price p_t^A then the investor who sells receives the amount

$$(p_t^A - c).$$

A buyer of the asset may also hold the asset to maturity, in which case she would receive the flow of dividends from time t to time T, and not incur the fee c at maturity. Let τ be a stopping time of the filtration $\{\mathcal{G}\}$. Write $\tau \geq t$ if a stopping time τ satisfies $t \leq \tau(\omega) \leq T$. Since every agent is risk-neutral and discounts the future at rate r, the maximum price that that an investor in group A is willing to pay at time $t < T$ given the maximum price process of B agents p_s^B, $s \geq t$ is:[9]

$$p_t^A = \sup_{\tau \geq t} \mathbb{E}^A \left\{ e^{-r(\tau - t)} (p_\tau^B - c\mathbf{1}_{\{\tau < T\}}) + \int_t^\tau e^{-r(u-t)} dD_u | \mathcal{G}_t \right\} \tag{2.5}$$

and similarly

$$p_t^B = \sup_{\tau \geq t} \mathbb{E}^B \left\{ e^{-r(\tau - t)} (p_\tau^A - c\mathbf{1}_{\{\tau < T\}}) + \int_t^\tau e^{-r(u-t)} dD_u | \mathcal{G}_t \right\} \tag{2.6}$$

with $p_T^A = p_T^B = 0$.

An equilibrium is thus a pair of processes (p_t^A, p_t^B) with $p_T^A = p_T^B = 0$, that satisfies equations (2.5) and (2.6).

[8] The assumption that the value of the asset at the terminal date is zero is not essential, what is needed is that at maturity the asset's value is common knowledge

[9] Here and in what follows the supremum over stopping times must be interpreted as an essential supremum, that is the smallest random variable that dominates a family of random variables (*e.g.* Karatzas and Shreve (1998), Appendix A).

It is natural to postulate that in our Markov environment there exists functions p^C, with

$$p_t^A = p^A(X_t, \hat{f}_t^A, t)) \quad \text{(resp.} \quad p_t^B = p^B(-X_t, \hat{f}_t^B, t))$$

We also guess a particular form for the functions p^C:

$$p^C(x, y, t) = \mathbb{E}_t^C[\int_t^T e^{-r(u-t)} \hat{f}_u^C du | \hat{f}_t^C = y] + q(x, t) \tag{2.7}$$

where $q \geq 0$ and non-decreasing in the difference of opinions. The first term on the right hand side of this expression measures the expected future dividends discounted at rate r and would be the price of the asset in a world with homogeneous expectations or when shorting is costless. Also since, $p^C(x, y, T) = 0$, $q(x, T) = 0$.

To derive an equation for q, we first observe that

$$\begin{aligned}
&p^A(X_t, \hat{f}_t^A, t) = \\
&\sup_{\tau \geq t} \mathbb{E}^A \left\{ e^{-r(\tau-t)}(p^B(-X_\tau, \hat{f}_\tau^A + X_\tau, \tau) - c\mathbf{1}_{\{\tau<T\}}) + \int_t^\tau e^{-r(u-t)} dD_u | \mathcal{G}_t \right\} = \\
&\sup_{\tau \geq t} \mathbb{E}^A \left\{ e^{-r(\tau-t)}(p^B(-X_\tau, \hat{f}_\tau^A + X_\tau, \tau) - c\mathbf{1}_{\{\tau<T\}}) + \int_t^\tau e^{-r(u-t)} \hat{f}_u^A du | \mathcal{G}_t \right\}
\end{aligned}$$

Here the last equality follows from the Law of Iterated Expectations and equation (2.1).

Using equations (2.4) and (2.7) we may write

$$\begin{aligned}
&p^A(X_t, \hat{f}_t^A, t) = \\
&\sup_{\tau \geq t} \mathbb{E}^A \Big\{ e^{-r(\tau-t)}(p^A(-X_\tau, \hat{f}_\tau^A, \tau) + \alpha(T-\tau, \lambda+r)X_\tau - c\mathbf{1}_{\{\tau<T\}}) + \\
&\int_t^\tau e^{-r(u-t)} \hat{f}_u^A du \,|\, \mathcal{G}_t \Big\}
\end{aligned} \tag{2.8}$$

The Law of Iterated Expectations extended to stopping times guarantees that for any $\tau \geq t$

$$\begin{aligned}
&\mathbb{E}^A \left[\int_t^\tau e^{-r(u-t)} \hat{f}_u^A du + \mathbb{E}^A(\int_\tau^T e^{-r(u-t)} \hat{f}_u^A du \,|\, \mathcal{G}_\tau) \,|\, \mathcal{G}_t \right] = \\
&\mathbb{E}^A \left[\mathbb{E}^A(\int_t^T e^{-r(u-t)} \hat{f}_u^A du \,|\, \mathcal{G}_\tau) \,|\, \mathcal{G}_t \right] = \mathbb{E}^A \left[\int_t^T e^{-r(u-t)} \hat{f}_u^A du \,|\, \mathcal{G}_t \right]
\end{aligned} \tag{2.9}$$

From expressions (2.7), (2.8) and (2.9), we obtain:

$$q^A(X_t, t) = \sup_{\tau \geq t} \mathbb{E}^A \left\{ e^{-r(\tau - t)} \left[\alpha(T - \tau, \lambda + r) X_\tau + q^A(-X_\tau, \tau) - c\mathbf{1}_{\{\tau < T\}} \right] \mid \mathcal{G}_t \right\} \tag{2.10}$$

$q^A(x, t)$ can be understood as the option value that A is willing to pay when the difference of opinions $\hat{f}^B - \hat{f}^A = x$, and there is $T - t$ to go. It satisfies an equation that resembles that of an American option, except that the right hand side involves again the option value.

Notice that the right hand side of (2.10) is potentially a function of ω that is measurable with respect to $\mathcal{G}_t$, while the left hand side only depends on X_t. It is however natural to expect that the supremum on the right hand side only depends on X_t. If we write $\{\mathcal{F}\}$ for the completed filtration generated by X, since, conditional on X_t, the past realizations of D and $\hat{f}$ do not help predict the value of X_s for $s > t$, we may choose a stopping time τ of the filtration $\{\mathcal{F}\}$ to solve the maximization problem (2.10).[10] Furthermore the Markov property of X guarantees that for each t, conditional on X_t the supremum on the right hand side does not depend on the past values of X_u, $u < t$.

Thus we may consider the problem: Find a function $q^A(x, t)$ such that if X_s solves (2.3) with $X_t = x$

$$q^A(x, t) = \sup_{\tau \geq t} \mathbb{E}^A \left\{ e^{-r(\tau - t)} \left[\alpha(T - \tau, \lambda + r) X_\tau + q^A(-X_\tau, \tau) - c\mathbf{1}_{\{\tau < T\}} \right] \middle| X_t = x \right\} \tag{2.11}$$

$$q^A(x, T) = 0 \tag{2.12}$$

and a symmetric problem involving q^B where $-X$ replaces X in the right hand side.

We call this set of equations the *Stochastic Formulation* of the option value. Establishing a solution to this problem is the focus of the next section.

[10] A detailed argument for this and the previous assertion would follow a similar result for discrete-time in Shiryaev (2007) (Theorem 21 on page 91), by discretizing the possible stopping times.

3 Direct solution of the stochastic formulation via a fixed point argument

We establish in this section the existence of a solution to the fixed-point problem described by (2.11) and(2.12).

If $X_s^{x,t}$, $t \le s \le T$ solves (2.3) with $X_t^{x,t} = x$ then

$$X_s^{x,t} = e^{-\rho(s-t)}x + \int_t^s \sigma e^{-\rho(s-u)} dW_u \tag{3.13}$$

By symmetry, we need to consider the problem only from the perspective of agents in group A and to lighten the notation we drop the superscript, writing $\mathbb{E}$ instead of $\mathbb{E}^A$, and q instead of q^A. In addition, we write $\alpha(u)$ in place of

$$\alpha(u, \lambda + r) = \frac{1 - e^{-u(\lambda+r)}}{\lambda + r}$$

and set $\alpha(u) = 0$ for $u < 0$.

3.1 Fixed point problem

Due to the boundary condition and the non-negativity of q we can restrict the search of solutions to the following set of functions:

$$H = \{h : \mathbb{R} \times (0, T] \to [0, +\infty) \text{ with } h(x, T) = 0\}.$$

For each $h \in H$ we define:

$$\mathbb{F}[h](x,t) = \tag{3.14}$$
$$\sup_{\tau \ge t} \mathbb{E}\left\{ e^{-r(\tau-t)} \left[h(-X_\tau, \tau) + \alpha(T-\tau)X_\tau - c1_{\{\tau<T\}} \right] \mid X_t = x \right\},$$

provided the right hand side is well defined. This is insured whenever the process

$$Y_s^h = e^{-r(T-s)} \left[h\left(-X_s, s\right) + \alpha\left(T-s\right) X_s - c1_{\{s<T\}} \right]$$

is right-continuous. Indeed, if we write

$$Z_t^h = \sup_{\tau \ge t} \mathbb{E}\left[Y^h(\tau) \mid X_t \right],$$

where the maximization is taken over all stopping times of the filtration $\{\mathcal{F}\}$, results in Appendix D in Karatzas and Shreve (1998) show that if Y^h

is right-continuous, Z^h is a super-martingale and that there exists a super-martingale $\tilde{Z}^h$, which is a cadlag modification of Z^h, that is $\tilde{Z}^h_t = Z^h_t$ a.s. for each $t \in [0,T]$.

Solving the equations implied by (2.11) and(2.12) in H is equivalent to finding a fixed point.

$$\mathbb{F}[h] = h \tag{3.15}$$

The map $\mathbb{F}$ is obviously monotone. If $h_1 \le h_2$ and $\mathbb{F}[h_1]$ and $\mathbb{F}[h_2]$ are well defined, then $\mathbb{F}[h_1] \le \mathbb{F}[h_2]$. We will show the existence of a subset of H where $\mathbb{F}$ is well defined and that is left invariant by $\mathbb{F}$. This will allow us to apply a monotone fixed point argument to insure the existence of a fixed point.

Write

$$S_1 = \{h \in H : \text{If } x \le y, 0 \le h(y,t) - h(x,t) \le (y-x)\alpha(T-t)\}.$$

For a given continuous function $\eta(x,t) > 0$ let

$$S_2^\eta = \{h \in H : \text{If } s > 0, h(x,t) - h(x,t+s) \le \eta(x,t)(\alpha(T-t) - \alpha(T-t-s))\},$$
$$S_3^\eta = \{h \in H : \text{If } s > 0, -\eta(-x,t)(\alpha(T-t) - \alpha(T-t-s)) \le h(x,t) - h(x,t+s)\}.$$

Lemma 3.1 *Assume $\mathbb{F}[h]$ is well defined and write $g = F[h]$. (a) If $h \in S_1$ then $g \in S_1$. (b) Set $\eta(x,t) = \frac{x}{2} + e^{\nu(T-t)}(1+x^2)$ with $\nu > 0$ large enough. If $h \in S_2^\eta$ then $g \in S_2^\eta$, and if $h \in S_3^\eta$ then $g \in S_3^\eta$.*

Proof: Preliminary note: Since $\mathbb{F}[\cdot](x,t)$ only depends on the distribution of $X^{x,t}_\cdot$, we sometimes replace $X^{t+r,x}_{s+r}$ by $X^{t,x}_s$, for $r \ge 0$.

If $h \ge 0$ then $g \ge 0$, since $\tau = T$ is always a feasible choice. Furthermore, since $h(\cdot,T) \equiv 0$, $g(x,T) = 0$ for each x. Thus if $h \in H$ then $g \in H$.

(a) Suppose that for $x \le y,\ 0 \le h(y,t) - h(x,t) \le (y-x)\alpha(T-t)$. Then,

$$\begin{aligned}
&g(y,t) - g(x,t) \le \\
&\sup_{\tau \ge t} \mathbb{E}\left[e^{-r(\tau-t)}\left(h(-X^{y,t}_\tau,\tau) - h(-X^{x,t}_\tau,\tau) + \alpha(T-\tau)(X^{y,t}_\tau - X^{x,t}_\tau)\right)\right] \le \\
&\sup_{\tau \ge t} \mathbb{E}\left[e^{-r(\tau-t)}\alpha(T-\tau)e^{-\rho(\tau-t)}(y-x)\right] \le \\
&\alpha(T-t)(y-x).
\end{aligned}$$

Here the second inequality follows from $-X^{y,t}_\tau \le -X^{x,t}_\tau$ and monotonicity of h, which also yields $g(y,t) - g(x,t) \ge 0$. The third inequality holds because each term under the conditional expectation decreases with τ. Furthermore,

$$\begin{aligned}
&g(y,t)-g(x,t)\geq \\
&\inf_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(h(-X_\tau^{y,t})-h(-X_\tau^{x,t})+\alpha(T-\tau)(X_\tau^{y,t}-X_\tau^{x,t})\right)\right] \\
\geq &\inf_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(\alpha(T-\tau)(X_\tau^{x,t}-X_\tau^{y,t})+\alpha(T-\tau)(X_\tau^{y,t}-X_\tau^{x,t})\right)\right] \\
&=0.
\end{aligned}$$

(b: Part I) We first consider an $h\in S_2^\eta$. Fix $s>0$ and write $\Delta\alpha(t)=\alpha(T-t)-\alpha(T-t-s)$. If $\tau\geq t$ is a stopping time, set $\tau(s)=\min\{\tau+s,T\}$. Then,

$$\begin{aligned}
&g(x,t)-g(x,t+s) \\
\leq\ &\sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(h(-X_\tau^{x,t},\tau)-h(-X_{\tau(s)}^{x,t+s}\tau(s))+X_\tau^{x,t}\Delta\alpha(\tau)-c\mathbf{1}_{\{\tau<T\}}+c\mathbf{1}_{\{\tau<T-s\}}\right)\right] \\
\leq\ &\sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(h(-X_\tau^{x,t},\tau)-h(-X_\tau^{x,t},\tau(s))+X_\tau^{x,t}\Delta\alpha(\tau)\right)\right] \\
\leq\ &\sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(\eta(-X_\tau^{x,t},\tau)(\alpha(T-\tau)-\alpha(T-\tau(s))+X_\tau^{x,t}\Delta\alpha(\tau)\right)\right] \\
=\ &\sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(\eta(-X_\tau^{x,t},\tau)\Delta\alpha(\tau)+X_\tau^{x,t}\Delta\alpha(\tau)\right)\right]
\end{aligned}$$

Here the first inequality follows since if $\tau(s)=T$, $h(\cdot,\tau(s))=0$ and $\alpha(T-\tau(s))=\alpha(T-\tau-s)=0$ (recall that we set $\alpha(u)=0$ for $u<0$.) The third inequality follows because $h\in S_2^\eta$, and the last equality holds again because when $\tau(s)=T$, $\alpha(T-\tau(s))=\alpha(T-\tau-s)=0$. We also replaced $X_{\tau(s)}^{x,t+s}$ by $X_\tau^{x,t}$, where appropriate (see note above).

Since

$$\eta(x,t)=\frac{x}{2}+e^{\nu(T-t)}(1+x^2),\ \nu>0.$$

$\eta(-x,t)+x=\eta(x,t)>0$. Since $\Delta\alpha>0$, and the exponential term $e^{-r(\tau-t)}\leq 1$, we can get the bound:

$$g(x,t)-g(x,t+s)\leq\mathbb{E}\eta(X_\tau^{x,t},\tau)\Delta\alpha(\tau)$$

Hence it suffices to show that $\eta(X_t,t)\Delta\alpha(t)$ is a super-martingale to insure that

$$g(x,t)-g(x,t+s)\leq\eta(x,t)(\alpha(T-t)-\alpha(T-t-s)) \tag{3.16}$$

If $t < T-s$, then $\Delta\alpha(t) = \kappa e^{(\lambda+r)t}$ where $\kappa = e^{-(\lambda+r)T}\frac{e^{(\lambda+r)s}-1}{\lambda+r}$. Ito's lemma guarantees that the drift of $\eta(X_t,t)\Delta\alpha(t)$ is

$$(\lambda+r-\rho)\frac{x}{2}-\frac{\rho}{2}-2\rho e^{\nu(T-t)}x^2+\sigma^2 e^{\nu(T-t)}+(\lambda+r-\nu)e^{\nu(T-t)}(1+x^2), \quad (3.17)$$

and if expression (3.17) is non-positive then $\eta(X_t,t)\Delta\alpha(t)$ is a super-martingale. Note that the $1 \le e^{\nu(T-t)} \le e^{\nu T}$. For large values of $|x|$, the quadratic terms dominates, which has coefficient $(\lambda+r)e^{\nu(T-t)} - \nu e^{\nu(T-t)} < 0$ as long as $\nu > \lambda + r$. However for a bounded set of $|x|'$s one can always choose ν large enough such that (3.17) holds. Hence, we can choose ν large enough to make $\eta(X_t,t)e^{(\lambda+r)t}$ a supermartingale as desired.

When $t \ge T-s$, then $\Delta\alpha(t) = \alpha(T-t)$ is decreasing, and so it is even easier to obtain a supermartingale. The same ν as above works, and so we have established the existence of a ν such that $\eta(X_t,t)\Delta\alpha(t)$ is a super-martingale.

(b: Part II) To show that if $h \in S_3^{\tilde{\eta}}$ then

$$g(x,t+s) - g(x,t) \ge -\eta(-x,t)(\alpha(T-t) - \alpha(T-t-s)), \quad (3.18)$$

consider a stopping time $\tau \ge t+s$. If $\tau < T$, set $\tau(s) = \tau - s$. If $\tau \ge T$ set $\tau(s) = T$. Note that $\mathbf{1}_{\{\tau(s)<T\}} \equiv \mathbf{1}_{\{\tau<T\}}$.

$$\begin{aligned}
& g(x,t+s) - g(x,t) \\
\le\ & \sup_{\tau\ge t+s} \mathbb{E}\left[e^{-r(\tau-t-s)}\left(h(-X_\tau^{x,t+s},\tau) - h(-X_{\tau(s)}^{x,t},\tau(s)) + X_{\tau(s)}^{x,t}\left(\alpha(T-\tau)-\alpha(T-\tau(s))\right)\right)\right] \\
\le\ & \sup_{\tau\ge t+s} \mathbb{E}\left[e^{-r(\tau-t-s)}\left(\eta(X_{\tau(s)}^{x,t},\tau(s)) - X_{\tau(s)}^{x,t}\right)\left(\alpha(T-\tau(s))-\alpha(T-\tau)\right)\right] \\
=\ & \sup_{\tau\ge t+s} \mathbb{E}\left[e^{-r(\tau-t-s)}\left(\eta(-X_{\tau(s)}^{x,t},\tau(s))\right)\left(\alpha(T-\tau(s))-\alpha(T-\tau)\right)\right]
\end{aligned}$$

The first inequality follows from the fact that if $\tau = T$ then $h(\cdot,\tau) = \alpha(T-\tau) = 0$. The second inequality holds because $h \in S_3^{\eta}$. We again replaced $X_\tau^{x,t+s}$ by $X_{\tau(s)}^{x,t}$, where appropriate (see note above).

Thus it again suffices to choose ν such that $\eta(-X_{\tau(s)}^{x,t},\tau(s))\Delta\alpha(\tau(s))$ is a supermartingale. But since $\eta(-X_\tau^{x,t},\tau) = \eta(X_\tau^{-x,t},\tau)$ (in law), the sub-martingale property of $\eta(-X_{\tau(s)}^{x,t},\tau(s))\Delta\alpha(\tau(s))$ for ν large follows immediately from the result in Part I above.

Choose the function η as in Lemma 3.1, ν large enough. Write

$$S = S_1 \bigcap S_2^{\eta} \bigcap S_2^{\eta}.$$

Every function $h \in S$ is locally Lipschitz continuous. In addition, for each $h \in S$ the process Y^h is right continuous since,

$$\lim_{s \searrow 0} [h(-X_{t+s}, t+s) - h(-X_t, t)] \leq$$
$$\lim_{s \searrow 0} [(\alpha(T-t-s)|X_{t+s} - X_t| + |h(-X_t, t+s) - h(-X_t, t)|] = 0,$$

because $h \in S_1 \bigcap S_2^\eta \bigcap S_3^\eta$ and X is continuous. Thus $\mathbb{F}[h]$ is well defined. Lemma 3.1 guarantees that for every $h \in S$, $\mathbb{F}[h] \in S$. The following result will be used in Section 5:

Lemma 3.2 *(i) For any compact $K \subset \mathbb{R}$, there exists a constant b_K such that for any $h \in S$, $x, y \in K$ and $s, t \in [0, T]$*

$$h(x,t) - h(y,s) \leq b_K \left(|x-y| + |t-s|\right).$$

(ii) There exists constant C_T such that for any $(x,t) \in \mathbb{R} \times [0,T]$

$$0 \leq h(x,t) \leq C_T \left(1 + \max\{0, x\}\right)$$

Proof: (i) is immediate. To prove (ii), notice that since $h \in S_1$,

$$h(x,t) \leq h(0,t) + \max\{0,x\}\ \alpha(T-t) \leq h(0,t) + \frac{\max\{0,x\}}{\lambda + r}.$$

Furthermore, from (i)

$$h(0,t) = h(0,t) - h(0,T) \leq b_0 T$$

Thus we get

$$0 \leq h(x,t) \leq b_0 T + \frac{\max\{0,x\}}{\lambda + r}.$$

3.2 Existence of a minimal solution to the fixed point

In this subsection we prove the following result:

Theorem 3.3 *There exists $q \in S$ which is a fixed point for $\mathbb{F}$. If $h \in H$ is any other fixed point of $\mathbb{F}$ then $q \leq h$, that is q is minimal.*

A *supersolution* to the fixed point problem (3.15) is a $\bar{h} \in H$ such that $\bar{h} \geq \mathbb{F}[\bar{h}]$. Our first result is:

Proposition 3.4 *There exists a supersolution to the fixed point problem (3.15).*

Proof: As in Lemma 3.1 we set $\eta(x,t) = \frac{x}{2} + e^{\nu(T-t)}(1+x^2),\ \nu > 0$. Let

$$h(x,t) = \eta(x,t)\alpha(T-t) + 1_{\{t<T\}}c.$$

Since $e^{(\lambda+r)s}$ is increasing in s the choice of ν in (b: Part I) in the proof of Lemma 3.1 guarantees that $\eta(X_t,t)$ is a super-martingale. Thus

$$(\mathbb{F}(h))(x,t) = \sup_{\tau\geq t}\mathbb{E}\left\{e^{-r(\tau-t)}\left[\eta(-X_\tau^{x,t},\tau)\alpha(T-\tau) + X_\tau^{x,t}\alpha(T-\tau)\right]\right\} =$$

$$\sup_{\tau\geq t}\mathbb{E}\left\{e^{-r(\tau-t)}\eta(X_\tau^{x,t},\tau)\alpha(T-\tau)\right\} = \eta(x,t)\alpha(T-t) = h(x,t) - 1_{\{t<T\}}c$$

Remark 3.5 *In fact there exists a continuous super-solution $\bar{q}$. Set*

$$\bar{q}(x,t) = \eta(x,t)\alpha(T-t).$$

$\bar{q}$ is continuous and by monotonicity,

$$\mathbb{F}[\bar{q}] \leq \mathbb{F}[h] = h - 1_{\{\tau<T\}}c = \bar{q}.$$

In particular this shows that when $c = 0$ there exists a continuum (indexed by ν) of continuous solutions to the fixed point problem.

Proof of Theorem 3.3: We adapt an argument used in the proof of Theorem 1 in Chen and Kohn (2011). We construct a sequence of functions:

$$\begin{cases} q_0 = 0, \\ \\ q_{n+1} := \mathbb{F}[q_n] \end{cases}$$

Since $q_0 \in S$, $q_n \in S$ for every n, and thus $\mathbb{F}[q_n]$ is well defined. Furthermore $q_1 \geq 0$ and thus monotonicity of $\mathbb{F}$ guarantees that $q_n \leq q_{n+1}$. Furthermore, $0 \leq \bar{h}$ and monotonicity implies $q_n \leq \mathbb{F}\bar{q} \leq \bar{h}$. Thus

$$q(x,t) := \lim_{n\to\infty} q_n = \sup_n q_n$$

is well defined and, since S is closed under pointwise convergence, $q \in S$. Then

$$\begin{gathered} q(x,t) = \sup_n q_{n+1}(x,t) = \sup_n \mathbb{F}[q_n](x,t) \\ = \sup_n \sup_{\tau\geq t} E\left[e^{-r(\tau-t)}\left(q_n(-X_\tau,\tau) + \alpha(T-\tau)X_\tau - c1_{\{\tau<T\}}\right)|X_t = x\right] \\ = \sup_{\tau\geq t}\sup_n \mathbb{E}\left[e^{-r(\tau-t)}\left(q_n(-X_\tau,\tau) + \alpha(T-\tau)X_\tau - c1_{\{\tau<T\}}\right)|X_t = x\right] \\ = \sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(\sup_n q_n(-X_\tau,\tau) + \alpha(T-\tau)X_\tau - c1_{\{\tau<T\}}\right)|X_t = x\right] \\ = \sup_{\tau\geq t}\mathbb{E}\left[e^{-r(\tau-t)}\left(q(-X_\tau,\tau) + \alpha(T-\tau)X_\tau - c1_{\{\tau<T\}}\right)|X_t = x\right] \\ = \mathbb{F}[q](x,t). \end{gathered}$$

Here we use the fact that we can always interchange the order of the supremums and that the monotone convergence theorem justifies interchanging supremum and conditional expectation.

In addition if $h \in H$ is any other fixed point of $\mathbb{F}$, monotonicity of this operator guarantees that $q_n \leq h$ and hence $q \leq h$.

4 Obstacle problem and viscosity solutions

We start with a heuristic derivation of a non-local obstacle problem that one should expect to be satisfied by a solution to the fixed point problem (3.15) and define the viscosity solutions to this obstacle problem.

If it is optimal to exercise the option to sell at $t < T$ when $X_t = x$ then

$$q(x,t) = \alpha(T-t)x + q(-x,t) - c.$$

If $t < T$ and there is no exercise, i.e. in the continuation region, we have

$$q(x,t) > \alpha(T-t)x + q(-x,t) - c$$

In the continuation region, the dynamic programming principle (at least formally) yields:

$$-q_t + \frac{1}{2}\sigma^2 q_{xx} - \rho x q_x - rq = 0.$$

In the stopping region, we get

$$-q_t + \frac{1}{2}\sigma^2 q_{xx} - \rho x q_x - rq \leq 0$$

Therefore, if we write the option value as a function of time to expiration, that is

$$u(t,x) = q(T-t,x), \text{ and writing}$$

$$\psi(x,t) := x\alpha(t) - c,$$

we get

$$\min\left\{u_t - (\frac{1}{2}\sigma^2 u_{xx} - \rho x u_x - ru), u(x,t) - (u(-x,t) + \psi(x,t))\right\} = 0.$$

Let

$$\mathcal{M}u = -\frac{1}{2}\sigma^2 u_{xx} + \rho x u_x + ru \text{ and } \mathcal{L}u = u_t + \mathcal{M}u$$

and $\mathcal{L}$ is thus a (forward) parabolic operator. Since $u(x,0) = q(x,T) = 0$, we obtain the following *obstacle problem*:

$$\begin{cases} u(x,0) = 0 \\ \min\left(\mathcal{L}u, \quad u(x,t) - (u(-x,t) + \psi(x,t))\right) = 0 \end{cases} \tag{4.19}$$

Scheinkman and Xiong (2003) showed that in the case of an infinite horizon, the value of the resale option is given by a solution to the following stationary problem, with $\psi_\infty(x) = \psi(x, +\infty) = \dfrac{x}{r+\lambda} - c$

$$\min\left(\mathcal{M}u_\infty, \quad u_\infty(x) - u_\infty(-x) - \psi_\infty(x)\right) = 0 \quad \text{for} \quad x \in \mathbb{R} \tag{4.20}$$

Obstacle problems such as (4.19) do not typically have classical solutions in the sense of a smooth function u that satisfies the equation. A particularly fruitful weaker notion of solution is that of a viscosity solution. To define a viscosity solution one starts with viscosity subsolutions and viscosity supersolutions. Although the definition given in Appendix A is necessarily technical to allow for non smooth functions, roughly a subsolution to (4.19) is a function u such that:

$$\begin{cases} u(x,0) \le 0 \\ \min\left(\mathcal{L}u, \quad u(x,t) - (u(-x,t) + \psi(x,t))\right) \le 0 \end{cases}$$

Similarly a supersolution can be thought as a function u such that

$$\begin{cases} u(x,0) \ge 0 \\ \min\left(\mathcal{L}u, \quad u(x,t) - (u(-x,t) + \psi(x,t))\right) \ge 0 \end{cases}$$

A viscosity solution is essentially a function u that is simultaneously a viscosity subsolution and a viscosity supersolution. See Appendix A for a precise definition. The concept of viscosity allows us to make sense of non-smooth solution u to the obstacle problem (4.19). It is well known that the value function associated with stochastic control problems is often non-smooth, that is not necessarily $C^{1,2}$. However it is often possible to show that the value function is a viscosity solution of the associated PDE.[11] In Section 5 we show that a similar result holds for our equilibrium problem - the equilibrium option value is an appropriate viscosity solution to the obstacle problem (4.19).

An important tool in the study of viscosity solutions is the comparison principle that states that if u is a subsolution and v a supersolution satisfying

[11] See *e.g.* Theorem VIIII.5.1 in Fleming and Soner (1993), Zariphopoulou (1994) or Bouchard (2007).

some growth conditions then $u \leq v$. The comparison principle is also an ideal tool for comparing solutions. Let $u(\gamma)$ be a family of solutions corresponding to values of a parameter γ. Suppose we show that if $\gamma < \gamma'$ then $u(\gamma)$ is a subsolution for the obstacle problem when the parameter value is γ'. Since $u(\gamma')$ is necessarily a supersolution for the obstacle problem for the parameter value γ', the comparison principle assures us that $u(\gamma) \leq u(\gamma')$. We exploit this idea repeatedly in Section 7.

Berestycki et al. (2014) establishes the following comparison principle.

Theorem 4.1 *If $c > 0$ and u (resp. v) is a subsolution (resp. supersolution) of (4.19) on $\mathbb{R} \times [0, T)$ for some $T > 0$, satisfying for some constant $C_T > 0$*

$$u(x,t) \leq C_T(1 + \max(0,x)) \quad \textit{and} \quad v(x,t) \geq -C_T(1 + \max(0,x))$$

for all $(x,t) \in \mathbb{R} \times [0,T)$. Then $u \leq v$ on $\mathbb{R} \times [0,T)$.

The formal proof is in Berestycki et al. (2014). A heuristic proof of this result follows: let u be a subsolution and v a supersolution and assume they are smooth and that

$$M = \sup(u - v) = (u - v)(x_0, t_0) > 0.$$

Then at the point (x_0, t_0):

$$\left\{ \begin{array}{ll} & i)\mathcal{L}u \leq 0 \\ \text{or} & \\ & ii) u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0) \leq 0 \end{array} \right. \tag{4.21}$$

$$\left\{ \begin{array}{ll} & \mathcal{L}v \geq 0 \\ \text{and} & \\ & v(x_0,t_0) - v(-x_0,t_0) - \psi(x_0,t_0) \geq 0 \end{array} \right. \tag{4.22}$$

i) If $\mathcal{L}u \leq 0$ then using $\mathcal{L}v \geq 0$, we obtain

$$0 \geq \mathcal{L}u - \mathcal{L}v = -\frac{1}{2}\sigma^2(u-v)_{xx} + r(u-v) > 0,$$

a contradiction.

ii) If, on the other hand, $\mathcal{L}u > 0$ then

$$u(x_0,t_0) - u(-x_0,t_0) - \psi(x_0,t_0) \leq 0 \tag{4.23}$$

Subtracting from (4.23) the second line of (4.22), we get

$$M = (u - v)(x_0, t_0) \leq (u - v)(-x_0, t_0) \leq M$$

Again case i) for $(-x_0, t_0)$ is straightforward, and it remains case ii) for $(-x_0, t_0)$, i.e.

$$u(-x_0, t_0) - u(x_0, t_0) - \psi(-x_0, t_0) \leq 0$$

Summing this inequality with (4.23), we get:

$$0 \geq -\psi(x_0, t_0) - \psi(-x_0, t_0) = 2c$$

a contradiction to $c > 0$.[12]

Using the comparison principle, Berestycki et al. (2014) establish an existence theorem for viscosity solutions of the obstacle problem (4.19). More precisely they establish the following theorem:

Theorem 4.2 *If* $c > 0$, *there exists a unique viscosity solution* u *of (4.19) satisfying*

$$|u - \max(0, \psi)| \leq C \quad on \quad \mathbb{R} \times [0, +\infty).$$

In Section 5 we will establish that the (minimal) equilibrium for the price of the resale option satisfies $q(x,t) = u(x, T-t)$ that is, except for a change in the time direction the minimal equilibrium price of the option is identical to the viscosity solution given in Theorem 4.2.

5 The dynamic programming principle and viscosity solutions

In this section we state the dynamic programming principle and the results connecting the solution to the fixed point problem (3.15) with the viscosity solution defined in Theorem 4.2. All proofs are left to Appendix B.

Theorem 5.1 *Suppose* $h \in H$ *is a solution of* $\mathbb{F}[h] = h$. *Then, for any stopping time* $\theta \geq t$

$$h(x,t) = \sup_{\tau \geq t} \mathbb{E} \Big\{ e^{-r(\theta - t)} h(X_\theta^{x,t}, \theta) 1_{\{\theta \leq \tau\}} + \\ e^{-r(\tau - t)} \left[h(-X_\tau^{x,t}, \tau) + \alpha(T - \tau) X_\tau^{x,t} - c 1_{\{\tau < T\}} \right] 1_{\{\theta > \tau\}} \Big\} \tag{5.24}$$

[12] Berestycki et al. (2014) shows that the comparison theorem is false for $c = 0$.

One way to get intuition for why Theorem 5.1 holds is to imagine the following modified optimization problem. The buyer of the asset at t is free to sell the asset to any other agent at any stopping time $\tau \geq t$. In that case, she must pay the trading cost c per unit of the asset that she sells. In addition, if a stopping time θ occurs while she still owns the asset, she is forced to sell to another agent of her own group, but is dispensed from paying the cost c. Theorem 5.24 states that the reservation price in this modified optimization problem is exactly the reservation price in the original problem. The reason why this result holds is that selling to someone in the same group yields exactly zero gains from trade.

The dynamic programming principle implies that $u(x,t) := q(x, T-t)$ solves the obstacle problem given by the system of equations (4.19). More precisely, we have:

Theorem 5.2 *(i) Let $h \in H$ be any solution of $\mathbb{F}[h] = h$. Then $(x,t) \mapsto h(x, T-t)$ is a viscosity solution of*

$$\min\left(\mathcal{L}u, \quad u(x,t) - (u(-x,t) + \psi(x,t))\right) = 0 \tag{5.25}$$

which satisfies for every x, $h_*(x,T) := \liminf_{(y,t)\to(x,T)} h(y,t) \geq 0$.
(ii) If moreover h satisfies

$$h^*(x,T) := \limsup_{(y,t)\to(x,T)} h(y,t) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}$$

then we have $h(x, T-t) = u(x,t)$ where u is a viscosity solution of (4.19).

As a consequence of this result and Lemma 3.2, we have

Corollary 5.3 *The minimal solution q of the stochastic formulation defined in Theorem 3.3 satisfies:*

$$q(x, T-t) = u(x,t)$$

where u is the unique viscosity solution of (4.19) that satisfies $|u - \max(0,\psi)| \leq C \quad on \quad \mathbb{R} \times [0, +\infty)$.

Remark 5.4 *To establish this Corollary it would suffice to prove Theorem 5.1, for continuous functions h. However by proving the general result for $h \in H$ we have actually shown that the minimal solution to the fixed-point problem q is the unique equilibrium option value among all functions $h \in H$ that satisfy the growth condition and such that $h^*(x,T) \leq 0$.*

6 Exercise boundary

Berestycki et al. (2014) shows that there exists a function $a : (0, +\infty) \to [0, +\infty)$, $a \in C^\infty(0, +\infty)$, such that for all $t > 0$:

$$\{x \geq 0 : u(x,t) - u(-x,t) = \psi(x,t)\} = \{x \geq a(t)\} \subset \left\{x \geq \frac{c}{\alpha(T-t)}\right\}$$

Since we showed that $q(\cdot, t) = u(\cdot, T-t)$ the function $k(t) = a(T-t)$ describes the *exercise boundary.* Agents in group A sell to group B agents at time t if $X(t) = k(t)$. Symmetrically, agents in group B sell at t if $X(t) = -k(t)$. Since $\alpha(T-t)k(t) \geq c$, agents in group A sell only when the expected value of discounted future differences in dividends exceed the fixed cost c. This result is intuitive once one considers the *gains from trade* that occur in a trade. If an agent in group A sells to an agent in group B at t when the difference of opinions is $x \geq 0$, the gains from trade are given by:

$$x\alpha(T-t) + q(-x,t) - q(x,t) \leq x\alpha(T-t),$$

since q is monotone in x. Since gains from trade must exceed the transaction cost c if a trade is to occur, $\alpha(T-t)k(t) \geq c$.

If a member of the B group buys the asset at t, she would pay a price that reflects her valuation of future dividends plus the value of the resale option she acquires, $q(-k(t), t)$. Symmetrically, if a member of the A group buys the asset at t, he would pay a price that reflects his valuation of future dividends plus the value of the resale option he acquires, $q(-k(t), t)$. Thus $q(-k(t), t)$ is in any case the speculative component of the buyer's reservation price. If a trade occurs at t we will define (the size of) a bubble as the speculative component at that trading time:

$$b(t) := q(-k(t), t) \equiv u(-a(T-t), T-t) \tag{6.26}$$

Notice that at a trading time the buyer is necessarily more optimistic than the seller, that is $k > 0$, but we do not include the difference in opinions as part of the bubble. In this sense our definition of a bubble is conservative.

7 Comparative statics

The monotonicity argument used in establishing the existence of the equilibrium option value q yields results on monotonicity of q and on the dependency of q on parameters:

Lemma 7.1 *The function q is increasing*[13] *and convex in x. Furthermore*

[13]In the results in this Section increasing should be read as non-decreasing etc...

q is decreasing in r, λ and c, and increasing in σ.

Proof: Since $q \in S$ the monotonicity with respect to x follows. Furthermore it is easy to check that the operator $\mathbb{F}$ preserves functions that are convex in x and since $q_0 \equiv 0$, q is convex. Monotonicity with respect to r, λ and c, follows from observing that the monotone map $\mathbb{F}$ is decreasing in r and λ (because of the expression for $]\alpha)$ and in c. Finally if we write $X_s^{\sigma,x,t}$ for the solution indexed by σ and $\sigma > \sigma'$, then $X_s^{\sigma,x,t}$ is a mean-preserving spread of $X_s^{\sigma',x,t}$. Thus for any convex h, $\mathbb{F}h$ must increase with σ. Since every $q_n = \mathbb{F}[q_{n-1}]$ is convex and $\mathbb{F}$ is increasing, q must increase with σ.

Monotonicity with respect to t and the monotonicity properties of the exercise boundary are not immediate consequences of properties of the operator $\mathbb{F}$. However Berestycki et al. (2014) use comparison principles to establish monotonicity properties of the viscosity solution u and the following result is a simple translation of Theorems 6.1 and 7.2 in Berestycki et al. (2014) using the relationship between u and q, and between k and a.

Lemma 7.2 *The minimal solution to the fixed point problem q is increasing in t. The exercise boundary k satisfies: (i) $k'(t) \geq 0$. (ii) A decrease in ρ increases k. (iii) An increase in c, λ, r or σ increases k. (iv) Let $k(t,c)$ be the exercise boundary as a function of time t and the transaction cost c. Set*

$$\bar{k}(t) = \left(\frac{3\sigma^2}{2(1+(\rho-\lambda)\alpha(T-t))}\right)^{1/3}.$$

Then $t < T$

$$\bar{k}(t) \leq \frac{k(t,c)}{c^{1/3}} \to \bar{k}(t)$$

uniformly in compact subsets of $[0,T)$, as $c \to 0$.

The occurrence of a trade at t depends on the particular realization of the Brownian W. Nevertheless, we can derive the following comparative statics results concerning $b(t)$ as immediate consequences of Lemmas 7.1 and 7.2.

Proposition 7.3 *(i) The function $b(t)$ is decreasing in t. (ii) A decrease in r or c increases b.*

The results in (ii) formalize the intuition that low interest rates and low transaction costs fuel bubbles. As the transaction cost $c \to 0$ the exercise boundary $k(t,c) \to 0$, for $t < T$. That is as $c \to 0$ trading takes place when difference of opinions and hence gains from trade are very small. Nonetheless the speculative value increases as c decreases. The intuition is that as $c \to 0$ the number of future trades becomes very large so even though the gain from trade at each time is close to zero, the accumulation of many future gains from trade yield a maximum value for the resale option.

Lemmas 7.1 and 7.2 state that an increase in σ increases the value of the option to resell and the value of the exercise boundary, what leaves the effect on b indeterminate. This difficulty can be surmounted by studying the obstacle problem that is solved by a rescaled version of the function u

$$\bar{u}_\sigma(y,t) = u(\sigma y, t),$$

and the corresponding rescaled exercise boundary

$$\bar{a}_\sigma(t) = \frac{a(t)}{\sigma}.$$

Applications of comparison principles to this rescaled problems yield:

Proposition 7.4 *An increase in σ increases b.*

Proof: See Appendix C.

It is well documented that bubbles occur in times of technical or financial innovations. It is reasonable to expect that during these times the volatility of differences of opinion increases. Proposition 7.4 shows that in our model bubbles do increase when difference in opinions are more volatile.

8 Effect of trading cost $c \sim 0$ on the time to next trade and the size of the bubble

The exercise boundary $k(t,c)$ depends on the cost of transaction c and converges to 0 as $c \to 0$, for any $t < T$. Thus, as $c \to 0$, one should expect trading volume in a given time interval to increase. In the infinite horizon case, Scheinkman and Xiong (2003) showed numerically that a small increase in trading costs near $c = 0$ affects much more the value of the exercise boundary than the value of the bubble. Here we will present some computations that help us understand this relationship. Notice that Lemma 7.2 guarantees that near $c = 0$, $k(t,c) \sim c^{1/3}\bar{k}(t)$. As a consequence, we show formally that the distribution of time between trades scales as $c^{2/3}$. In particular the elasticity of the median (or any quantile) time between trades with respect to c is -2/3. Thus a 1% increase in c, caused perhaps by a "Tobin tax" on transactions, would decrease the median time between trades by $\frac{2}{3}$ of 1%, if c is small.

In contrast, the value of the bubble b is much less sensitive to the cost of trading c. We provide a formal argument that the elasticity of b with respect to c converges to 0 as $c \to 0$. When c is small, a 1% increase in c would have a negligible effect on the size of the bubble. Thus a Tobin tax is much more effective in reducing trade than in reducing bubbles.

Consider first the distribution of times between trades. This distribution concerns the minimum time τ it takes to reach $X_{t+s} = k(t+s, c)$ starting from $X_t = -k(t, c)$. Fix $h > 0$, $\epsilon > 0$ and $t > 0$. The uniform convergence guaranteed by Lemma 7.2 insures that for small $c > 0$, we have $\bar{k}(t+s)c^{1/3} \leq k(t+s, c) \leq (\bar{k}(t+s) + \epsilon)c^{1/3}$ for $0 \leq s \leq hc^{2/3}$. So, if we define $\tau(z)$ as the first time we hit a curve $z(t+s)$, then:

$$\tau(c^{1/3}\bar{k}(\cdot)) \leq \tau(k(\cdot, c)) \leq \tau((\bar{k}(\cdot) + \epsilon)c^{1/3})$$

For small enough c, we have $|\bar{k}(t+s) - \bar{k}(t)| < \epsilon$ for $s < hc^{2/3}$, and hence, on the event that $\tau(k) - s < hc^{2/3}$, we can bound $\tau(k(\cdot, c))$ by:

$$\tau_1 := \tau(k_1 c^{1/3}) \leq \tau(k(\cdot, c)) \leq \tau(k_2 c^{1/3}) =: \tau_2$$

where $k_1(t+s) := \bar{k}(t) - \epsilon$, and $k_2(t+s) := \bar{k}(t) + 2\epsilon$ are constant functions.

We will now approximate $P(\tau_i - t < hc^{2/3})$ for $i = 1, 2$, and show that the approximations converge to each other as c and $\epsilon \to 0$.

First we show that, in the limit, we may set $\rho = 0$. More precisely let R be the first time after t that $X_{t+s} = \pm 1$. Then $P(R < t + hc^{2/3}) \to 0$ as $c \to 0$. So, given any $\epsilon > 0$ for small enough c, with probability at least $1 - \epsilon$, $|X_{t+s}| < 1$ for $s \leq hc^{2/3}$, and hence the drift term in X_{t+s} has size at most $hc^{2/3}\rho < \epsilon c^{1/3}$ for small c. So $|X_{t+s} - \sigma W_{t+s}| < \epsilon c^{1/3}$ for $s < hc^{2/3}$.

Recall that the time $\tau*$ it takes for a Brownian Motion σW to go from level $-a$ to level $+b$ is given by:

$$P(\tau* < s) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{b+a}{\sigma\sqrt{s}}}^{\frac{b+a}{\sigma\sqrt{s}}} e^{-x^2/2} dx$$

Thus combining these approximation results, we get the bound:

$$P(\tau_1 - t < hc^{2/3}) \geq \frac{1}{\sqrt{2\pi}} \int_{-\kappa}^{\kappa} e^{-x^2/2} dx - \epsilon$$

where

$$\kappa = \frac{k(t, c)c^{-1/3} + \bar{k}(t) - 2\epsilon}{\sigma\sqrt{h}}.$$

In the definition of κ, $k(t), cc^{-1/3}$ comes from the value of the initial location (all other terms in numerator and denominator had a $c^{1/3}$ term, so this gets factored out). The $\bar{k}(t) - \epsilon$ is from the definition of τ_1 (and hence comes from the fact that $\bar{k}$ is continuous). A second $-\epsilon$ comes from the comparison of X_{t+s} with σW_{t+s}. Finally, the $-\epsilon$ after the integral comes from the fact that this comparison might fail on a set of probability ϵ. But since $\epsilon > 0$ was arbitrary, and $k(t,c)c^{-1/3} \to \bar{k}(t)$ as $c \to 0$, we can get the asymptotic bound:

$$\lim_{c} P(\tau_1 - t < hc^{2/3}) \geq \frac{1}{\sqrt{2\pi}} \int_{-\kappa'}^{\kappa'} e^{-x^2/2} dx$$

where $\kappa' = \frac{2\bar{k}(t)}{\sigma\sqrt{h}}$ as desired. The bound for τ_2 follows the same argument, and so the limit for τ follows.

The formal argument concerning the value of the bubble goes as follows. To make explicit the dependency on the trading cost c, set $b(t,c) := q(-k(t,c), t, c)$. Thus formally:

$$\frac{d\ln b}{d\ln c} = \frac{c\left(-q_x \frac{\partial k}{\partial c} + q_c\right)}{q} = \frac{-kq_x \frac{d\ln k}{d\ln c} + cq_c}{q}.$$

Lemma 7.1 states that $\frac{d\ln k}{d\ln c} = \frac{2}{3}$ and $k \sim 0$ for c small. Since $q \in S_1$, q_x is uniformly bounded and since q increases as $c \to 0$ we have that

$$\lim_{c\to 0} \frac{d\ln b}{d\ln c} = \lim_{c\to 0} \frac{cq_c}{q}.$$

In Appendix D we show formally that $q_c = O(c^{-1/3})$ and thus that $\lim_{c\to 0} \frac{d\ln b}{d\ln c} = 0$.

9 Concluding remarks

We have characterized the equilibrium value of the resale option on a finitely lived asset, when investors agree to disagree. The model is capable of generating the correlations observed between bubbles and trading frenzies and may be used to evaluate the impact of transaction taxes. The approach combines a fixed-point problem that arises from a stochastic formulation - whose solution uses martingale methods - with a PDE approach. The use of this dual approach yields comparison of solution results that are not obvious even in the stationary context of Scheinkman and Xiong (2003) where analytical solutions to the value of the resale option are available.

Missing are considerations of supply of the asset. It has been observed that bubbles often come to an end because supply of the over-valued asset increases.[14] To consider the effect of supply would require introducing risk-averse agents or limits to capital. In both cases we would have to increase the dimensionality of the problem, what lies beyond the scope of this paper.

Appendix

A Viscosity solutions

The following definitions of viscosity solutions proposed by Barles and Perthame (1987) are used in Berestycki et al. (2014):

Definition A.1 *(Viscosity sub/super/solution of equation* (4.19)*) Let* $T \in (0, +\infty]$.
(i) Viscosity sub/supersolution on $\mathbb{R} \times (0, T)$
A function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ *is a viscosity subsolution (resp. supersolution) of* (4.19) *on* $\mathbb{R} \times (0, T)$*, (that is, of the second equation in* (4.19)*), if* u *is upper semi-continuous (resp. lower semi-continuous), and if for any function* $\varphi \in C^{2,1}(\mathbb{R} \times (0, T))$ *and any point* $P_0 = (x_0, t_0) \in \mathbb{R} \times (0, T)$ *such that* $u(P_0) = \varphi(P_0)$ *and*

$$u \leq \varphi \quad \textit{on} \quad \mathbb{R} \times (0, T) \qquad (\textit{resp.} \quad u \geq \varphi \quad \textit{on} \quad \mathbb{R} \times (0, T))$$

then

$$\min \{(\mathcal{L}\varphi)(x_0, t_0), \quad u(x_0, t_0) - u(-x_0, t_0) - \psi(x_0, t_0)\} \leq 0,$$

$$(\textit{resp.} \qquad \min \{(\mathcal{L}\varphi)(x_0, t_0), \quad u(x_0, t_0) - u(-x_0, t_0) - \psi(x_0, t_0)\} \geq 0).$$

(ii) Viscosity sub/supersolution on $\mathbb{R} \times [0, T)$
A function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ *is a viscosity subsolution (resp. supersolution) of* (4.19) *on* $\mathbb{R} \times [0, T)$*, (that is, of the initial value problem), if* u *is a viscosity subsolution (resp. supersolution) of* (4.19) *on* $\mathbb{R} \times (0, T)$ *and satisfies moreover* $u(x, 0) \leq 0$ *(resp.* $u(x, 0) \geq 0$*) for all* $x \in \mathbb{R}$.

(iii) Viscosity solution on $\mathbb{R} \times [0, T)$
A function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ *is a viscosity solution of* (4.19) *on* $\mathbb{R} \times [0, T)$,

[14] See Scheinkman (2014) Section 2.2 for examples and references.

if and only if u^* is a viscosity subsolution and u_* is a viscosity supersolution on $\mathbb{R} \times [0,T)$ where

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(y,s) \qquad \textit{and} \qquad u_*(x,t) = \liminf_{(y,s)\to(x,t)} u(y,s).$$

B Dynamic programming principle and necessity condition for the viscosity solution: Proof

Proof of Theorem 5.1 : The proof follows closely Bouchard (2007).
Step 1: preliminaries
Setting

$$K^{x,t}(s) := e^{-r(s-t)}\left[h(-X_s^{x,t}, s) + \alpha(T-\sigma)X_s^{x,t} - c1_{\{s<T\}}\right]$$

we can rewrite the condition $\mathbb{F}(h) = h$ as

$$h(x,t) = \sup_{\tau \geq t} \mathbb{E}\left\{K^{x,t}(\tau)\right\}$$

From the flow property of stopping times:

$$X_s^{x,t} = X_s^{X_\theta^{x,t},\theta} \quad \text{for all} \quad s \in [\theta, T], \quad \mathcal{P}\text{-a.s.}$$

If $\theta \leq s \leq T$, then

$$K^{x,t}(s) = e^{-r(s-t)}\left[h(-X_s^{X_\theta^{x,t},\theta}, s) + \alpha(T-s)X_s^{X_\theta^{x,t},\theta} - c1_{\{s<T\}}\right] = e^{-r(\theta-t)}K_{X_{x,t}(\theta),\theta}(s)$$

This shows that

$$K^{x,t}(\tau) = e^{-r(\theta-t)}K^{X_\theta^{x,t},\theta}(\tau)1_{\{\theta\leq\tau\}} + K^{x,t}(\tau)1_{\{\theta>\tau\}}$$

Step 2: $\leq$ holds in (5.24)
Using the law of iterated expectantions, we get

$$\mathbb{E}\left\{K^{x,t}(\tau)\right\} = \mathbb{E}\left\{K^{x,t}(\tau)1_{\{\theta>\tau\}} + e^{-r(\theta-t)}1_{\{\theta\leq\tau\}}\mathbb{E}\left\{K^{X_\theta^{x,t},\theta}(\tau) \mid (X_\theta^{x,t}, \theta)\right\}\right\} \tag{2.27}$$

where we have used the fact that $\tau \geq \theta$ $\mathcal{P}$-a.s. in the second expectation. In particular

$$\mathbb{E}\left\{K^{x,t}(\tau)\right\} \leq \mathbb{E}\left\{K^{x,t}(\tau)1_{\{\theta>\tau\}} + e^{-r(\theta-t)}1_{\{\theta\leq\tau\}}h(X_\theta^{x,t},\theta)\right\} =$$

$$\mathbb{E}\left\{e^{-r(\theta-t)}h(X_\theta^{x,t},\theta)1_{\{\theta\leq\tau\}} + e^{-r(\tau-t)}\left[h(-X_\tau^{x,t},\tau) + \alpha(T-\tau)X_\tau^{x,t} - c1_{\{\tau<T\}}\right]1_{\{\theta>\tau\}}\right\}$$

because for any fixed stopping time $\bar{\theta} \geq t$, $\{\tau \geq t\} \bigcap \{\tau \geq \bar{\theta}\} = \{\tau \geq \bar{\theta}\}$. Taking the supremum on $\tau \geq t$ gives the desired inequality.

Step 3: $\geq$ holds in (5.24)

Fix $\varepsilon > 0$. Then for any $(y, s) \in \mathbb{R} \times (-\infty, T]$, there exists a stopping time $\tilde{\tau}_\varepsilon = \tilde{\tau}_\varepsilon(y, s) \geq s$ such that

$$\mathbb{E}\left\{K^{y,s}(\tilde{\tau}_\varepsilon)\right\} \geq -\varepsilon + h(y, s)$$

Choose a stopping time $\theta \geq t$, and set Then we set

$$\tau_\varepsilon = \tilde{\tau}_\varepsilon(X^{x,t}(\theta), \theta)$$

which is also a stopping time satisfying $\theta \leq \tau_\varepsilon$, $\mathcal{P}$-a.s. This implies that

$$\mathbb{E}\left\{K^{X_\theta^{x,t},\theta}(\tau_\varepsilon) \mid (X_\theta^{x,t}, \theta)\right\} \geq -\varepsilon + h(X_\theta^{x,t}, \theta)$$

Given a stopping time $\tau \geq t$, we now consider

$$\bar{\tau} = \tau 1_{\{\theta > \tau\}} + \tau_\varepsilon 1_{\{\theta \leq \tau\}}$$

which is also stopping time. Then (2.27) implies

$$\begin{aligned} \mathbb{E}\left\{K^{x,t}(\tau)\right\} &\geq \mathbb{E}\left\{K^{x,t}(\tau) 1_{\{\theta > \tau\}} + e^{-r(\theta - t)} 1_{\{\theta \leq \tau\}} \left\{-\varepsilon + h(X_\theta^{x,t}, \theta)\right\}\right\} \\ &\geq -\varepsilon + \mathbb{E}\left\{K_{x,t}(\tau) 1_{\{\theta > \tau\}} + e^{-r(\theta - t)} 1_{\{\theta \leq \tau\}} h(X_\theta^{x,t}, \theta)\right\} \end{aligned}$$

Taking the supremum over $\tau \geq t$,

$$h(x, t) \geq -\varepsilon + \sup_{\tau \geq t} \mathbb{E}\left\{e^{-r(\theta - t)} h(X_\theta^{x,t}, \theta) 1_{\{\theta \leq \tau\}} + e^{-r(\tau - t)} \left[h(-X_\tau^{x,t}, \tau) + \alpha(T - \tau) X_\tau^{x,t} - c 1_{\{\tau < T\}}\right] 1_{\{\theta > \tau\}}\right\}$$

and because $\varepsilon > 0$ is arbitrary, we obtain the desired inequality.

Proof of Theorem 5.2.: Again we follow closely Bouchard (2007).

Let

$$v(x, t) := h(x, T - t).$$

and recall that $h \geq 0$, and thus $v_*(x, 0) = h_*(x, T) \geq 0$, which shows that v satisfies the initial condition $v(x, 0) = 0$ in the viscosity sense for supersolutions.

We now want to show that v is a viscosity solution of (5.25). For $(x,t) \in \mathbb{R} \times (-\infty, T)$, since $\tau = t$ is always a possible choice,

$$h(x,t) \geq h(-x,t) + \alpha(T-t)x - c$$

That is,

$$v(x,t) - v(-x,t) - \psi(x,t) \geq 0 \tag{2.28}$$

Step 1: proof that v_* is a supersolution of (5.25)
Inequality (2.28) implies in particular (because ψ is continuous)

$$v_*(x,t) - v_*(-x,t) - \psi_*(x,t) \geq 0$$

Therefore, in order to show that v_* is a supersolution of (5.25), it remains to show in the viscosity sense that (since $\mathcal{L}v = v_t + \mathcal{M}v$)

$$v_t + \mathcal{M}v \geq 0 \quad \text{on} \quad \mathbb{R} \times (0, +\infty) \tag{2.29}$$

Let φ be a test function satisfying

$$h_* \geq \varphi \quad \text{with equality at} \quad P_0 = (x_0, t_0) \quad \text{with} \quad t_0 < T$$

Assume by contradiction that v_* is not a supersolution at $(x_0, T - t_0)$, i.e. (taking into account the change of sign in the time derivative in (2.29), because of the inversion of the time direction from h to v):

$$-\varphi_t + \mathcal{M}\varphi < 0 \quad \text{at} \quad P_0$$

As usual, for $P = (x,t)$, replacing, if necessary, $\varphi(P)$ by $\varphi(P) - |P - P_0|^4$, we can assume that there exists $\delta, \eta > 0$ small such that

$$\begin{cases} -\varphi_t + \mathcal{M}\varphi < 0 & \text{on} \quad B_\delta(P_0) \subset \mathbb{R} \times (-\infty, T), \\ h_* \geq \varphi + \eta & \text{on} \quad \partial B_\delta(P_0), \\ h_* = \varphi & \text{at} \quad P_0 \end{cases} \tag{2.30}$$

Let us consider a sequence of points $P_n = (x_n, t_n)$ for $n \geq 1$, such that

$$(P_n, h(P_n)) \to (P_0, h_*(P_0))$$

Now we define for each $n \geq 1$:

$$\theta_n = \inf \left\{ T \geq t \geq t_n, \quad (X_t^{P_n}, t) \notin B_\delta(P_0) \right\}$$

which is a stopping time $\geq t_n$]. We then choose $\tau = \theta_n$ in the dynamic programming principle (5.24) which yields

$$\begin{aligned} h(x_n, t_n) & \geq \mathbb{E}\left\{e^{-r(\theta_n - t_n)} h(X^{P_n}_{\theta_n}, \theta_n)\right\} \\ & \geq \mathbb{E}\left\{e^{-r(\theta_n - t_n)} h_*(X^{P_n}_{\theta_n}, \theta_n)\right\} \\ & \geq \mathbb{E}\left\{e^{-r(\theta_n - t_n)} \left\{\eta + \varphi(X^{P_n}_{\theta_n}, \theta_n)\right\}\right\} \\ & \geq e^{-2r\delta}\eta + \mathbb{E}\left\{e^{-r(\theta_n - t_n)} \varphi(X^{P_n}_{\theta_n}, \theta_n)\right\} \end{aligned}$$

where in the third line, we have used the second line of (2.30). We then use Ito's formula with stopping times to get for

$$\Phi(y, s) = e^{-r(s-t_n)}\varphi(y, s) \tag{2.31}$$

(up to redefining Φ outside the ball $B_\delta(P_0)$ such that Φ_x is bounded):

$$\begin{aligned} & \mathbb{E}\left\{e^{-r(\theta_n - t_n)} \varphi(X^{P_n}_{\theta_n}, \theta_n)\right\} \\ & = \mathbb{E}\left\{\Phi(X^{P_n}_{\theta_n}, \theta_n)\right\} \\ & = \Phi(x_n, t_n) + \mathbb{E}\left\{\int_{t_n}^{\theta_n} \left\{\Phi_t(X^{P_n}_s, s) + \tfrac{1}{2}\sigma^2 \Phi_{xx}(X^{P_n}_s, s) - \rho X^{P_n}_s \Phi_x(X^{P_n}_s, s)\right\}\ ds\right\} \\ & = \Phi(x_n, t_n) + \mathbb{E}\left\{\int_{t_n}^{\theta_n} e^{-r(s-t_n)} \left\{(\varphi_t - \mathcal{M}\varphi)(X^{P_n}_s, s)\right\}\ ds\right\} \\ & \geq \Phi(x_n, t_n) = \varphi(x_n, t_n) \end{aligned}$$

where in the last line, we have used the first line of (2.30).
Therefore

$$h(x_n, t_n) \geq e^{-2r\delta}\eta + \varphi(x_n, t_n)$$

Passing to the limit in n, we get

$$h_*(P_0) \geq e^{-2r\delta}\eta + \varphi(P_0)$$

Contradiction with the last line of (2.30).
This shows that v_* is a supersolution of (5.25).
Step 2: proof that v^* is a subsolution of (5.25)
Recall that (2.28) holds , which implies

$$v^*(x, t) - v^*(-x, t) - \psi(x, t) \geq 0$$

Therefore v^* is a subsolution if and only if we show that

$$v_t + \mathcal{M}v \leq 0 \quad \text{on} \quad \{(x, t) \in \mathbb{R} \times (0, +\infty),\ v^*(x, t) - v^*(-x, t) - \psi(x, t) > 0\} \tag{2.32}$$

Let φ be a test function satisfying

$$h^* \leq \varphi \quad \text{with equality at} \quad P_0 = (x_0, t_0) \quad \text{with} \quad t_0 < T$$

and

$$h^*(x_0, t_0) > h^*(-x_0, t_0) + \psi(x_0, T - t_0)$$

Assume by contradiction that v^* is not a subsolution at $(x_0, T - t_0)$, i.e. (taking into account the change of sign in the time derivative in (2.29), because of the inversion of the time direction from h to v):

$$-\varphi_t + \mathcal{M}\varphi > 0 \quad \text{at} \quad P_0$$

As usual, for $P = (x, t)$, up to replacing $\varphi(P)$ by $\varphi(P) + |P - P_0|^4$, we can assume that there exists $\delta, \eta > 0$ small such that

$$\begin{cases} -\varphi_t + \mathcal{M}\varphi > 0 & \text{on} \quad B_\delta(P_0) \subset \mathbb{R} \times (-\infty, T), \\ \varphi(x,t) \geq \eta + h^*(-x,t) + 1_{\{\tau < T\}}\psi(x, T-t) & \text{on} \quad B_\delta(P_0) \subset \mathbb{R} \times (-\infty, T), \\ \varphi \geq \eta + h^* & \text{on} \quad \partial B_\delta(P_0), \\ \varphi = h^* & \text{at} \quad P_0 \end{cases} \tag{2.33}$$

Consider a sequence of points $P_n = (x_n, t_n)$ for $n \geq 1$, such that

$$(P_n, h(P_n)) \to (P_0, h^*(P_0))$$

Now we define for each $n \geq 1$:

$$\theta_n = \inf \left\{ t \geq t_n, \quad (X_t^{P_n}, t) \notin B_\delta(P_0) \right\}$$

which is a stopping time $\geq t_n$. Given $\tau \geq t_n$, let (using Φ defined in (2.31))

$$I_n(\tau) = \mathbb{E}\left\{\Phi(X_{P_n}(\theta_n \wedge \tau), \theta_n \wedge \tau)\right\}$$

Applying Ito's formula with stopping time $\theta_n \wedge \tau$ we get

$$\begin{aligned} & I_n(\tau) \\ & = \Phi(x_n, t_n) + \mathbb{E}\left\{ \int_{t_n}^{\theta_n \wedge \tau} \left\{ \Phi_t(X_s^{P_n}, s) + \tfrac{1}{2}\sigma^2 \Phi_{xx}(X_s^{P_n}, s) - \rho X_s^{P_n} \Phi_x(X_s^{P_n}, s) \right\} \, ds \right\} \\ & = \Phi(x_n, t_n) + \mathbb{E}\left\{ \int_{t_n}^{\theta_n \wedge \tau} e^{-r(s - t_n)} \left\{ (\varphi_t - \mathcal{M}\varphi)(X_s^{P_n}, s) \right\} \, ds \right\} \\ & \leq \Phi(x_n, t_n) = \varphi(x_n, t_n) \end{aligned}$$

where in the last line, we have used the first line of (2.33). On the other hand, we have

$$
\begin{aligned}
& I_n(\tau) \\
& = \mathbb{E}\left\{1_{\{\tau<\theta_n\}}\Phi(X_\tau^{P_n},\tau) + 1_{\{\tau\geq\theta_n\}}\Phi(X_{\theta_n}^{P_n},\theta_n)\right\} \\
& = \mathbb{E}\left\{1_{\{\tau<\theta_n\}}e^{-r(\tau-t_n)}\varphi(X_\tau^{P_n},\tau) + 1_{\{\tau\geq\theta_n\}}e^{-r(\theta_n-t_n)}\varphi(X_{\theta_n}^{P_n},\theta_n)\right\} \\
& \geq \mathbb{E}\left\{e^{-r(\tau-t_n)}\left[1_{\{\tau<\theta_n\}}\left\{\eta + h^*(-X_\tau^{P_n},\tau) + 1_{\{\tau<T\}}\psi(X_\tau^{P_n},T-\tau)\right\} + 1_{\{\tau\geq\theta_n\}}\left\{\eta + h^*(X_{\theta_n}^{P_n},\theta_n)\right\}\right]\right\} \\
& \geq e^{-2r\delta}\eta + \mathbb{E}\left\{1_{\{\tau<\theta_n\}}e^{-r(\tau-t_n)}h(-X_\tau^{P_n},\tau) + 1_{\{\tau<T\}}\psi(X_\tau^{P_n},T-\tau) + 1_{\{\tau\geq\theta_n\}}e^{-r(\theta_n-t_n)}h(X_{\theta_n}^{P_n},\theta_n)\right\}
\end{aligned}
$$

where in the fourth line, we have used the second and third lines of (2.33). This implies that

$$
\begin{aligned}
& \varphi(x_n,t_n) \geq e^{-2r\delta}\eta + \\
& \mathbb{E}\left\{1_{\{\tau<\theta_n\}}e^{-r(\tau-t_n)}h(-X_\tau^{P_n},\tau) + 1_{\{\tau<T\}}\psi(X_\tau^{P_n},T-\tau) + 1_{\{\tau\geq\theta_n\}}e^{-r(\theta_n-t_n)}h(X_{\theta_n}^{P_n},\theta_n)\right\}
\end{aligned}
$$

Passing to the supremum on τ and using the dynamic programming principle (Theorem 5.1) we get

$$\varphi(x_n,t_n) \geq e^{-2r\delta}\eta + h(x_n,t_n)$$

Passing to the limit in n, we get

$$\varphi(P_0) \geq e^{-2r\delta}\eta + h^*(P_0),$$

a contradiction to the last line of (2.33).
This shows that v^* is a subsolution of (5.25) and ends the proof of the theorem.

Proof of Corollary 5.3
We set

$$\check{u}(x,t) = q(x,T-t)$$

Since q is continuous and $q(x,T)=0$ $q^*(x,T)=0$. Thus Theorem 4.2 implies that $\check{u}$ is a viscosity solution of the system of equations (4.19). In addition, both $\check{u}$ and u satisfy the growth condition of the Comparison Principle of Berestycki et al. (2014) (Theorem 4.1 above). Consequently we have $\check{u} \leq u$ and $\check{u} \geq u$.

C Proof of Proposition 7.4

To make explicit the dependence of $\mathcal{L}$ on σ write

$$\mathcal{L}_\sigma u(x,t) := u_t - \frac{\sigma^2}{2} u_{xx} + \rho x u_x + r u$$

and

$$\psi_\sigma(y,t) := \sigma y \alpha(t) - c$$

Step 1: preliminaries: Berestycki et al. (2014) show that the antisymmetric part of u

$$w(x,t) := u(x,t) - u(-x,t)$$

is the unique viscosity solution to a problem related to the obstacle problem that u solves, namely:

$$\begin{cases} \min(\mathcal{L}_\sigma w, w - \psi) = 0 & \text{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ w(0,t) = 0 & \text{for} \quad t \in (0,+\infty), \\ w(x,0) = 0 & \text{for} \quad x \in (0,+\infty) \end{cases} \tag{3.34}$$

that satisfies for some C

$$|w(x,t)| \leq C\,(1 + |x|) \tag{3.35}$$

and that a comparison principle holds for sub/super solutions that satisfy (3.35).

In addition, the function

$$\tilde{w}(x,t) := w(x,t) - \psi(x,t)$$

is the unique viscosity solution that satisfies for some C

$$|\tilde{w}(x,t)| \leq C\,(1 + |x|)\,, \tag{3.36}$$

of the problem:

$$\begin{cases} \min(\mathcal{L}_\sigma \tilde{w} + f, \tilde{w}) = 0 & \text{for} \quad (x,t) \in (0,+\infty) \times (0,+\infty), \\ \tilde{w}(0,t) = c & \text{for} \quad t \in (0,+\infty), \\ \tilde{w}(x,0) = c & \text{for} \quad x \in (0,+\infty) \end{cases} \tag{3.37}$$

Furthermore, a comparison principle holds for sub/super solutions of (3.37) that satisfy (3.36).

The function a can be alternatively described as:

$$\{x \geq a(t)\} = \{x \geq 0 : w(x,t) = \psi(x,t)\} = \{x \geq 0 : \tilde{w}(x,t) = 0\}\,.$$

Step 2: the $\bar{u}_\sigma$ problem
Set

$$y = \frac{x}{\sigma} \text{ and } \bar{u}_\sigma(y,t) = u(\sigma y, t).$$

Recall that u solves

$$\begin{cases} \min\left(\mathcal{L}_\sigma u, u(x,t) - u(-x,t) - \psi_1(x,t)\right) = 0 & \text{for} \quad (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 0 & \text{for} \quad x \in \mathbb{R} \end{cases}$$

Thus $\bar{u}_\sigma$ solves problem $\mathcal{P}(\sigma)$

$$\begin{cases} \min\left(\mathcal{L}_1 \bar{u}, \bar{u}_\sigma(y,t) - \bar{u}_\sigma(-y,t) - \psi_\sigma(y,t)\right) = 0 & \text{for} \quad (y,t) \in \mathbb{R} \times (0,+\infty), \\ u(y,0) = 0 & \text{for} \quad y \in \mathbb{R} \end{cases}$$

We also know that $\bar{u}_\sigma$ must solve the following problem

$$\begin{cases} \min\left(\mathcal{L}_1 \bar{u}_\sigma, \bar{u}_\sigma(y,t) - \bar{u}_\sigma(-y,t) - \psi_\sigma(y,t)\right) = 0 & \text{for} \quad (y,t) \in (0,+\infty) \times (0,+\infty), \\ \mathcal{L}_1 \bar{u}_\sigma = 0 & \text{for} \quad (y,t) \in \left(-\infty, \dfrac{c}{\alpha(+\infty)}\right) \times (0,+\infty), \\ \bar{u}_\sigma(y,0) = 0 & \text{for} \quad y \in \mathbb{R} \end{cases}$$

Since $\frac{\partial \psi_\sigma}{\partial \sigma} > 0$ whenever $y > 0$, if $\sigma' \geq \sigma$ then $\bar{u}_\sigma$ is a subsolution for the problem $\mathcal{P}(\sigma)$. However $\mathcal{P}(\sigma)$ is very similar to problem (4.19) and the proof of the comparison principle in Berestycki et al. (2014) can be easily adapted. Thus $\bar{u}_\sigma$ is weakly increasing in σ.

Step 3: the $\tilde{\bar{w}}$ problem
Defining

$$\bar{a}_\sigma(t) = \frac{a(t)}{\sigma}, \quad \bar{w}_\sigma(y,t) = \bar{u}_\sigma(y,t) - \bar{u}_\sigma(-y,t)$$

we see that $\bar{w}_\sigma$ solves

$$\begin{cases} \min\left(\mathcal{L}_1 \bar{w}_\sigma, \bar{w}_\sigma - \psi_\sigma\right) = 0 & \text{for} \quad (y,t) \in \Omega := (0,+\infty) \times (0,+\infty), \\ \bar{w}_\sigma = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and thus

$$\tilde{\bar{w}}_\sigma = \bar{w}_\sigma - \psi_\sigma$$

solves problem $\tilde{\mathcal{P}}_\sigma$

$$\begin{cases} \min\left(\mathcal{L}_1\tilde{\tilde{w}}_\sigma + \mathcal{L}_1\psi_\sigma, \tilde{\tilde{w}}\right) = 0 & \text{for} \quad (y,t) \in \Omega := (0,+\infty)\times(0,+\infty), \\ \tilde{\tilde{w}}_\sigma = c & \text{on} \quad \partial\Omega \end{cases}$$

with

$$\{(y,t)\in\Omega, \quad \tilde{\tilde{w}}_\sigma(y,t)=0\} = \{(y,t)\in\Omega, \quad y\geq \bar{a}(t)\}$$

A computation shows that

$$\mathcal{L}_1\psi_\sigma(y,t) = \sigma y\alpha' + (\rho+r)\sigma y\alpha - rc = \sigma y\alpha\left\{\frac{\alpha'}{\alpha}+\rho+r\right\} - rc$$

Thus if $\sigma' \geq \sigma$ then $\tilde{\tilde{w}}_\sigma$ is a super solution for the problem $\tilde{\mathcal{P}}_{\sigma'}$. Problem $\tilde{\mathcal{P}}_\sigma$ is very similar to (3.37) and the proof of the comparison principle in Berestycki et al. (2014) for (3.37) applies. The comparison principle implies that $\tilde{\tilde{w}}_\sigma \geq \tilde{\tilde{w}}_{\sigma'}$. Theorems 7.1 (ii) in Berestycki et al. (2014) guarantees that $\frac{\partial w_\sigma}{\partial y} \leq \sigma\alpha(t)$ (in the distribution or viscosity sense) and thus $\tilde{\tilde{w}}_\sigma$ is non-increasing in y. Hence $\bar{a}_\sigma$ is (weakly) decreasing in σ. Since $\bar{u}$ is increasing in y,

$$b(t) = \bar{u}(-\bar{a}(T-t), T-t)$$

is weakly increasing in σ.

D Formal Asymptotics as $c \to 0$

We first establish that

$$q_c = O\left(\frac{1}{c^{\frac{1}{3}}}\right)$$

Recall that $q(x,t) = u(x,T-t)$ so it suffices to prove that $u_c = O\left(\frac{1}{c^{\frac{1}{3}}}\right)$ As in Appendix C, we write

$$\tilde{w}(x,t) = w(x,t) - \psi(x,t), \quad w(x,t) = u(x,t) - u(-x,t).$$

Proposition 9.5 of Berestycki et al. (2014) establishes that for $c>0$ small and $x \geq 0$,

$$\tilde{w}(x,t) = cv^0(y,t) \quad \text{with} \quad y = \frac{x}{c^{\frac{1}{3}}} \quad \text{and} \quad v^0(y,t) \simeq \phi(\bar{y}) \quad \text{with} \quad \bar{y} = \frac{y}{\bar{a}(t)}$$

where

$$\phi(\bar{y}) = \begin{cases} \dfrac{\bar{y}^3}{2} - \dfrac{3}{2}\bar{y} + 1 & \text{if} \quad 0 \le \bar{y} \le 1, \\ \\ 0 & \text{if} \quad \bar{y} > 1 \end{cases}$$

and

$$\bar{a}(t) = \left(\frac{3\sigma^2}{2(1 + (\rho - \lambda)\alpha(t))} \right)^{\frac{1}{3}}$$

Therefore formally:

$$\tilde{w}_c(x,t) = \phi(\bar{y}) - \frac{1}{3}\bar{y}\phi'(\bar{y}) =: \bar{\phi}(\bar{y}) = (1 - \bar{y})_+$$

and

$$\tilde{w}_c(x,t) = u_c(x,t) - u_c(-x,t) + 1$$

This implies that for $x \ge 0$:

$$(\partial_x u_c)(x,t) + (\partial_x u_c)(-x,t) = \frac{1}{c^{\frac{1}{3}}\bar{a}(t)} \bar{\phi}'(\bar{y}) = -\frac{1}{c^{\frac{1}{3}}\bar{a}(t)} \, 1_{[0,1]}(\bar{y})$$

i.e.

$$2(\partial_x u_c)(0,t) = -\frac{1}{c^{\frac{1}{3}}\bar{a}(t)}$$

Hence if we set

$$z^c := c^{\frac{1}{3}} u_c$$

Then z^c solves:

$$\begin{cases} \mathcal{L} z^c = 0 & \text{in} \quad \{x < a(t)\}, \\ \partial_x z^c = c^{\frac{1}{3}} (\partial_x u_c)(x,t) & \text{for} \quad x = a(t), \\ z^c(x,0) = 0 & \text{for} \quad x \in \mathbb{R} \end{cases}$$

For $c = 0$, this gives formally:

$$\begin{cases} \mathcal{L} z^0 = 0 & \text{in} \quad \{x < 0\}, \\ \partial_x z^0 = -\dfrac{1}{2\bar{a}(t)} & \text{for} \quad x = 0, \\ z^0(x,0) = 0 & \text{for} \quad x < 0 \end{cases}$$

and then

$$z^0 = O(1)$$

which implies the formal result.

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